Geometric Asian Option Pricing in General Affine Stochastic Volatility Models with Jumps

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Abstract

In this paper we present some results on Geometric Asian option valuation for affine stochastic volatility models with jumps. We shall provide a general framework into which several different valuation problems based on some average process can be cast, and we shall obtain close-form solutions for some relevant affine model classes.

Keywords: Geometric Asian Options, Average Strike Options, Average Price Options, Stochastic Volatility, Affine Processes.

1 Introduction

Asian options are contingent claims exhibiting an explicit dependence on some average of the underlying asset process on a specified time interval, which usually coincides with the option lifetime. They can be classified according to their payoff structure.

If S_T is the value of the underlying asset at maturity T, K is the strike price, and A_T is a suitably defined average of the values assumed by the stock during the period under consideration, the "Average Strike" (also called "Floating Strike") Asian call payoff is provided by the following expression: $(S_T - A_T)_+$, while the payoff of the "Average Price" (Sometimes called "Fixed Strike" or "Average Rate") Asian call is given by: $(A_T - K)_+$.

Different kind of averages can be considered in order to define the Asian options payoff: the arithmetic, geometric and sometimes harmonic average are in use. The main interest on this kind of options from a financial point of view is due to the appealing property that the averaging procedure can "smoothing out" the underlying price process behavior, by offering a protection against sudden dramatic changes and making hedging strategies more feasible. The interest on these options from the mathematical point of view is mainly due to the challenge represented by

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the description of the joint stochastic dynamics of the underlying price process and its average. This interest can justify the quite large amount of literature focused on the subject.

We want here to recall some of the main results available on Asian options pricing, but the list we are going to provide is certainly far to be exhaustive. Let's start by considering the results on Arithmetic Asian options, i.e. those options for which the average under consideration is the arithmetic average. When the underlying price dynamics is described by a Geometric Brownian motion a possible approach to the valuation problem is presented in the papers by H.Geman, M. Yor [GY93] and by D. Dufresne [Duf01]. In a more recent paper the same author [Duf05] provides some more explicit valuation formulas based on properties of the Bessel process. The paper by M. Schröder [Sch08] offers some more results on Arithmetic Asian options again on complex analysis methods. When the underlying dynamics is described by an exponential Lévy process, the valuation problem for Arithmetic Asian options is investigated in the papers by H. Albrecher [Alb04] and by H. Albrecher and M. Predota [AP04]. An efficient approximation methodology for the cases in which close-formulas are not available is proposed in the paper by Milevsky [MP98]. As far as diffusion type dynamics for the underlying are concerned, including stochastic volatility, we mention here the paper by J.-P. Fouque and C.-H. Han [FH03]. By moving to the more general case of the underlying dynamics described by a semimartingale process very few results are available. We just mention the paper by J. Večer and M. Xu [VX04], based on a partial integro-differential equation (PIDE) approach. When explicit valuation formulas are not available, sharp lower and upper bounds intervals for option prices can be useful in improving the quality of the approximations adopted: some results along this direction are provided in the papers by S. Simon, M.J. Goovaerts, and J. Dhaene for Arithmetic Asian options [SGD00].

For Geometric Asian Options a straightforward calculation provides an explicit solution for the Average Price call valuation problem, while only a numerical the Average Strike call value can be obtained only by numerical approximations [WHD95]. As far as the exponential Lévy models are concerned, we just mention the paper by C.B. Zhang and C.W. Oosterlee [ZO13].

For local volatility models, the paper by I. Peng [Pen06] deals with the Geometric Asian options valuation problem for the CEV model. Y.L. Cheung and H.Y. Wong [CW04b] provide some semi-analytical formulas for Geometric Asian options in stochastic volatility models exhibiting a mean-reverting behavior. More recent results, based on asymptotic expansion techniques, have been obtained by E. Gobet and M. Miri [GM14] and in a very recent paper by B. Kim and I.-S. Wee [KW11] the Geometric Asian option pricing problem has been studied for the stochastic volatility model proposed by S.L. Heston.

We mention here that in financial practice it is quite common to define the averages appearing in the option payoff on a discrete monitoring basis rather than continuous. The average defined in a continuous monitoring framework can be then considered only a mathematical idealization and approximation of the true object involved in the financial model. Just to recall some basic results available in the discrete monitoring setting, for Asian options valuation in exponential Lévy framework, we mention the paper by G. Fusai and A. Meucci [FM08] and the paper by M. Vanmaele, G. Deelstra, J. Liinev, J. Dhaene and M.J. Goovaerts [VDJLG06] providing lower and upper bounds for Arithmetic Asian options.

The introduction of jumps *or* stochastic volatility in the underlying asset dynamics can remove only part of the drawbacks provided by a description based on Geometric Brownian motion. In particular it is well known that exponential Lévy models exhibit reasonable implied volatility smiles only for short maturities, while stochastic volatility models produce more realistic smiles for long maturities. In the attempt to get implied volatility smiles more realistic on the whole maturity spectrum new model classes have been proposed, in which both features are present. We mention here some of the most popular models of this kind. The model suggested by D. Bates [Bat96] combines the features of the jump-diffusion model proposed by R. Merton [Mer76] with those of the stochastic volatility model proposed by Heston [Hes93]. O.E. Barndorff-Nielsen and N. Shephard [BNS01], [BNNS02] introduced a model in which the volatility dynamics is described by an Ornstein-Uhlenbeck process driven by a subordinator. The models proposed by P. Carr, H. Geman, D. Madan and M. Yor [CGMY03], [CW04a], sometimes called Time-Changed Lévy models, represent also a reasonable attempt to improve the asset price dynamics description. We mention also the model introduced by D. Bates [Bat00], in which an affine (deterministic) dependence is included between the stochastic volatility and the jumps intensity; we are going to denote this model as "Turbo-Bates" in the rest of the paper, in order to avoid confusion with the model proposed previously by the same author.

All the above mentioned pricing models in which stochastic volatility features have been combined with jumps belong to the large family of the so-called affine models, according to the definition provided by D. Duffie, D. Filipovic and W. Schachermayer [DFS03]. This class includes almost all the most popular pricing models existing in the literature related to many different type of underlying assets: fixed income securities, credit risk models, equities and commodities. Many relevant features of these models can be described in a unified way by the very general framework provided by the affine process approach. For an extensive treatment of the general properties of affine models and some related technical issues we mention the Thesis by M. Keller-Ressel [KR08].

We want to recall here another relevant class of valuation problems requiring the description of some average process, i.e. options on realized volatility and variance swaps. Several recent papers attacked these valuation problems in different setting. In the paper by J. Kallsen, J. Muhle-Karbe and M. Voss [KMKV11] the pricing of options on variance in affine stochastic volatility models has been extensively investigated; some results in a Barndorff-Nielsen and Shephard modeling framework were provided in [BGK07], while in [CLW12] variance swaps pricing has been studied for time-changed Lévy models.

In [HS11] a semi-explicit evaluation formula for Geometric Asian Options, for fixed and floating strike, under continuous monitoring, when both stochastic volatility and jumps come into play has been provided; in that paper a specific model framework was considered, i.e. the Barndorff-Nielsen and Shephard model.

The contribution of the present paper is to develop a general valuation scheme and to provide some semi-explicit evaluation formulas for Geometric Asian Options, when the underlying process describing the joint dynamics of logreturns and volatility is affine. We shall provide a quite general framework into which several different valuation problems can be formulated and solved: all those based on the geometric mean of the variables, including average price and average strike Asian call options; variance options valuation can be also cast into the present framework, but only for continuous return processes, as we shall discuss in Section 3.

In next section we shall introduce the general setting and the notations used throughout the paper, while in Section 3 we shall present the introductory results on the affine representation for integral functionals. After providing in Section 4 an auxiliary result based on a change-of-numeraire technique, which will turn out to be useful for Average Strike option calculations, in Section 5 we shall present the general results on Geometric Asian options valuation in a general affine framework. In Section 6 we shall apply our general results to the most popular concrete affine stochastic volatility models and we shall provide the semi-explicit formulas for both Average Price and Average Strike options for these models. In section 7 we shall resume the main results obtained in this paper and we'll outline some possible developments of the present work.

2 Model Setup

The purpose of this section is to clarify the framework in which we are going to develop our pricing problem and to clarify the basic notations adopted in the following. The results recalled here are mainly based on the treatment provided in [KR08] and [KR11]. We fix some time horizon $T > 0$ up to which we wish to model the price process of some financial asset. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space, which supports all the processes we encounter in the sequel.

We shall call an affine process a stochastically continuous, time-homogeneous Markov process $(X_t, \mathbb{P}^x)$ with state space $D = \mathbb{R}_+^m \times \mathbb{R}^n$ if its characteristic function is an exponentially affine function of the state vector, i.e., if there exist functions $\phi : \mathbb{R}_+ \times \mathcal{U} \to \mathbb{C}_-$, $\boldsymbol{\psi} : \mathbb{R}_+ \times \mathcal{U} \to \mathcal{U}$ such that

$$\log\big(\mathbb{E}[\exp \boldsymbol{u} \cdot \boldsymbol{X}_t | \boldsymbol{X}_0]\big) = \phi(t, \boldsymbol{u}) + \boldsymbol{X}_0 \cdot \boldsymbol{\psi}(t, \boldsymbol{u}) \tag{1}$$

for all $(t, \boldsymbol{u}) \in \mathbb{R}_+ \times \mathcal{U}$ and where

$$\mathbb{C}_- := \{u \in \mathbb{C} : \Re u \le 0\} \quad \text{and} \quad \mathcal{U} := \mathbb{C}_-^m \times i\mathbb{R}^n. \tag{2}$$

By convention, the logarithm above denotes the distinguished logarithm in complex plane, that makes ϕ and ψ jointly continuous in the complex plane (cf. [DFS03]). Note that due to the Markov property an analogous equation also holds true for expectations conditional on $\boldsymbol{X}_s$, that is

$$\log\big(\mathbb{E}[\exp \boldsymbol{u} \cdot \boldsymbol{X}_t | \boldsymbol{X}_s]\big) = \phi(t-s, \boldsymbol{u}) + \boldsymbol{X}_s \cdot \boldsymbol{\psi}(t-s, \boldsymbol{u}) \tag{3}$$

for all $0 \le s \le t$ and $\boldsymbol{u} \in \mathcal{U}$. An affine process is called **regular** if the derivatives:

$$F(\boldsymbol{u}) := \frac{\partial \phi}{\partial t}(t, \boldsymbol{u})|_{t=0^+}, \ \mathbf{R}(\boldsymbol{u}) := \frac{\partial \boldsymbol{\psi}}{\partial t}(t, \boldsymbol{u})|_{t=0^+},$$

exist for all $\boldsymbol{u} \in \mathcal{U}$, and are continuous at $\boldsymbol{u} = 0$. It has been shown in [KRST11] that *any* affine process in the sense of the above definition is regular and hence that the functions $F(\boldsymbol{u})$ and $R(\boldsymbol{u})$ are well-defined.

Since the functions $F(u)$ and $\boldsymbol{R}(\boldsymbol{u})$ completely characterize the process $(\boldsymbol{X}_t)_{t\ge0}$ they are called the **functional characteristics of** $(\boldsymbol{X}_t)_{t\ge0}$.

In the following we shall need the notion of **truncation function**, but we'll specify which truncation function will be used whenever it will be necessary. When an affine process will be assumed to describe the price dynamics of some asset, we shall refer to it as an affine pricing model.

In the following we shall assume the (risk-neutral) stock price process S_t to be given as

$$S_t = \exp\big\{(r-q)t + X_t\big\}, \tag{4}$$

where r is the risk-free interest rate, q is the dividend yield and X_t is the discounted dividend-corrected log-price process.

Let V_t denote another (one-dimensional) process with $V_0 > 0$, such that (X_t, V_t) is a stochastically continuous, time-homogeneous Markov process.

We define the process (X_t, V_t) an Affine Stochastic Volatility model if the cumulant generating function of (X_t, V_t) is of the special affine form

$$\log\big(\mathbb{E}[\exp\{uX_t + wV_t\} | X_0, V_0]\big) = \phi(t, u, w) + V_0 \psi(t, u, w) + X_0 u. \tag{5}$$

Note that this setup is as in [KR11, Section 5], from where we will adopt the nomenclature and call (X_t, V_t) affine stochastic volatility (ASV) process and the associated asset price model ASV model.

Remark 1. *A bivariate affine model has functional characteristics F and $\boldsymbol{R} = (R_1, R_2)$. An ASV has $R_1 = 0$ and we set $R = R_2$ and call simply F, R the functional characteristics.*

The following theorem characterizes regular ASV processes and provides a representation result for the functions F, R.

Theorem 1. *[DFS03, Theorem 2.7] Let $(X_t, V_t)_{t\geq 0}$ be a regular ASV process. Then there exist a set of parameters $(a, \alpha, b, \beta, c, \gamma, m, \mu)$ where a, α are positive semi-definite matrices, $b, \beta \in \mathbb{R}^2$, $c, \gamma \geq 0$ and m, μ are Lévy measures on $\mathbb{R}^2$, such that*

$$F(u,w) = \frac{1}{2}(u,w)\cdot a \cdot (u,w)^T + b\cdot (u,w)^T - c + \int_{D\setminus\{0\}} \left(e^{xu+yw} - 1 - h_F(x,y)\cdot (u,w)^T\right) m(dx,dy)$$

$$R(u,w) = \frac{1}{2}(u,w)\cdot \alpha \cdot (u,w)^T + \beta\cdot (u,w)^T - \gamma + \int_{D\setminus\{0\}} \left(e^{xu+yw} - 1 - h_R(x,y)\cdot (u,w)^T\right) \mu(dx,dy),$$

holds, where $h_F(x,y)$, $h_R(x,y)$ are suitable truncation functions. Furthermore the functions ϕ and ψ in (5) fulfill the generalized Riccati equations:

$$\begin{aligned} \partial_t\phi(t,u,w) &= F(u,\psi(t,u,w)), & \phi(0,u,w) &= 0, \\ \partial_t\psi(t,u,w) &= R(u,\psi(t,u,w)), & \psi(0,u,w) &= w. \end{aligned} \tag{6}$$

For option pricing we employ a structure preserving martingale measure. This means, we choose an equivalent martingale measure, such that the model structure remains unchanged, only model parameters change. For several particular models enjoying the affine structure, a systematic investigation has been performed on the class of equivalent martingale measure, also providing a full characterization of the subclass of structure preserving measures: for the BNS model we mention the paper by E. Nicolato and E. Venardos [NV03], while for the Bates model a brief discussion on the subject is included in [Bat96].

The following proposition provides a sufficient condition for an affine process to be conservative (i.e. non-exploding) and a martingale:

Proposition 1. *[KR11, Corollary 2.1] Let (X_t, V_t) be defined as before and the quantity $\chi(u)$ be defined as follows:*

$$\chi(u) := \frac{\partial R}{\partial w}(u,w)|_{w=0}. \tag{7}$$

If $F(0,0) = R(0,0) = F(1,0) = R(1,0) = 0$ and $\max\{\chi(0), \chi(1)\} < \infty$, then $\exp\{X_t\}$ is a conservative process and a martingale.

Note that $F(0,0) = R(0,0) = 0$ is equivalent to $c = \gamma = 0$.

3 Integral functionals for ASV models

Our starting point is an affine ASV model (X, V) as introduced above. To study Geometric Asian options or realized variance options we introduce the associated integral processes Y and Z with

$$Y_t = \int_0^t X_s ds, \qquad Z_t = \int_0^t V_s ds. \tag{8}$$

Proposition 2. *If (X, V) is an ASV model with functional characteristics (F, R), then the joint law of (X_t, V_t, Y_t, Z_t) is described by*

$$\log E[e^{u_1X_t+u_2V_t+u_3Y_t+u_4Z_t}|X_0, V_0] = \Phi(t,u_1,u_2,u_3,u_4) + (u_1+u_3t)X_0 + \Psi(t,u_1,u_2,u_3,u_4)V_0 \tag{9}$$

where

$$\dot{\Phi} = F(u_1 + u_3 t, \Psi) \qquad \Phi(0) = 0 \tag{10}$$
$$\dot{\Psi} = R(u_1 + u_3 t, \Psi) + u_4 \qquad \Psi(0) = u_2. \tag{11}$$

Proof. It follows from [KR08, Theorem 4.10, p.50] for two dimensions, that (X, V, Y, Z) is affine,

$$\log E[e^{u_1 X_t + u_2 V_t + u_3 Y_t + u_4 Z_t} | X_0, V_0, Y_0, Z_0] = \Phi(t) + \psi_1(t) X_0 + \psi_2(t) V_0 + \psi_3(t) Y_0 + \psi_4(t) Z_0, \tag{12}$$

where the Φ and ψ_i satisfy the Riccati equations

$$\dot{\Phi} = F(\psi_1, \psi_2) \qquad \Phi(0) = 0 \tag{13}$$
$$\dot{\psi}_1 = \psi_3 \qquad \psi_1(0) = u_1 \tag{14}$$
$$\dot{\psi}_2 = R(\psi_1, \psi_2) + \psi_4 \qquad \psi_2(0) = u_2 \tag{15}$$
$$\dot{\psi}_3 = 0 \qquad \psi_3(0) = u_3 \tag{16}$$
$$\dot{\psi}_4 = 0 \qquad \psi_4(0) = u_4. \tag{17}$$

Remember that the solutions of (13) depend on the parameters u_1, u_2, u_3, u_4, thus $\psi_1(t) = \psi_1(t; u_1, u_2, u_3, u_4)$ etc. Some of those equations can be immediately integrated. Obviously $\psi_3(t) = u_3$, $\psi_4(t) = u_4$, $\psi_1(t) = u_1 + u_3 t$ and the only relevant equations are

$$\dot{\Phi} = F(u_1 + u_3 t, \psi_2) \qquad \Phi(0) = 0 \tag{18}$$
$$\dot{\psi}_2 = R(u_1 + u_3 t, \psi_2) + u_4 \qquad \psi_2(0) = u_2 \tag{19}$$

Then we note that $Y_0 = 0$ and $Z_0 = 0$ and we set $\Psi = \psi_2$. (36)and (37) follow from (18) and (19). ☐

Remark 2. *Variance swaps and options on realized variance in stochastic volatility models with jumps have been studied in [BGK07] and [Sep08]. In a general affine setting they have been investigated extensively in the paper [KMKV11] where the realized variance is approximated by the quadratic variation of the log-return process. For continuous return processes, such as the Heston model, for example, the quadratic variation $[X, X]$ and its predictable part $\langle X, X \rangle$ coincide with integrated variance, which is our Z.*

The cumulant of the integrated variance can be computed according to the following corollary, which turns out to be a special case of [KMKV11, Lemma 5.1, P.634].

Corollary 1.

$$\log E[e^{w Z_t}] = \phi(t, w) + V_0 \psi(t, w) \tag{20}$$

where

$$\dot{\phi} = F(0, \psi) \qquad \phi(0) = 0 \tag{21}$$
$$\dot{\psi} = R(0, \psi) + w \qquad \psi(0) = 0 \tag{22}$$

Proof. This follows immediately from Prop.2 with $u_1 = 0$, $u_2 = 0$, $u_3 = 0$, $u_4 = w$. ☐

4 Change of numeraire for ASV models

To calculate the price of the average strike option we apply the change-of-numeraire technique and take the stock as a new numeraire.

From now on we denote the martingale measures with the bond resp. stock as a numeraire by Q^0 resp. Q^1, and expectations E^0 resp. E^1.

$$\log E^0_{x,v}[e^{u_1 X(t)+u_2 V(t)}] = \phi^0(t,u_1,u_2) + x\psi^0_1(t,u_1,u_2) + v\psi^0_2(t,u_1,u_2) \tag{23}$$

Thus we have the density process

$$\frac{dQ^1}{dQ^0}(t) = e^{X_t - x} \tag{24}$$

on $\mathcal{F}_t$.

Let's start with the following

Lemma 1. *If (X,V) is affine under Q^0 with functional characteristics F^0 and R^0, then it is affine under Q^1 with functional characteristics F^1 and R^1 given by*

$$F^1(u_1,u_2) = F^0(u_1+1,u_2), \quad R^1(u_1,u_2) = R^0(u_1+1,u_2) \tag{25}$$

Proof.

$$\begin{aligned}\log E^1_{x,v}[e^{u_1 X(t)+u_2 V(t)}] &= \log E^0_{x,v}[e^{x+X_t}\cdot e^{u_1 X(t)+u_2 V(t)}] = \\ &- x + \log E^0_{x,v}[e^{(u_1+1)X(t)+u_2 V(t)}] = \\ \phi^0(t,u_1+1,u_2) &+ x(\psi^0_1(t,u_1+1,u_2)-1) + v\psi^0_2(t,u_1+1,u_2) = \\ &\phi^1(t,u_1,u_2) + x\psi^1_1(t,u_1,u_2) + v\psi^1_2(t,u_1,u_2)\end{aligned} \tag{26}$$

with

$$\phi^1(t,u_1,u_2) = \phi^0(t,u_1+1,u_2), \tag{27}$$
$$\psi^1_1(t,u_1,u_2) = \psi^0_1(t,u_1+1,u_2) - 1, \tag{28}$$
$$\psi^1_2(t,u_1,u_2) = \psi^0_2(t,u_1+1,u_2) \tag{29}$$

Thus

$$F^1(u_1,u_2) = F^0(u_1+1,u_2) \quad R^1(u_1,u_2) = R^0(u_1+1,u_2), \tag{30}$$

□

If e^X is a martingale we have $F^0(1,0) = R^0(1,0) = 0$ and thus $F^1(0,0) = R^1(0,0) = 0$.

Lemma 2. *If (X,V) is an ASV model, then the joint law of (X_t,Y_t) under Q^1 is described by*

$$\log E^1[e^{uX_t+wY_t}] = \phi^1(t,u,w) + v\psi^1(t,u,w) \tag{31}$$

where

$$(\phi^1)' = F(u+1,\psi^1) \qquad \phi^1(0) = 0 \tag{32}$$
$$(\psi^1)' = R(u+1,\psi^1) \qquad \psi^1(0) = w. \tag{33}$$

Proof. This follows from Lemma 1 and Proposition 2 applied to Q^1 resp. F^1, R^1. □

5 General results for Geometric Asian options

5.1 Average price

Let us denote by $\bar{X}_T$ the arithmetic average of the log-returns process and by $\hat{S}_T$ the geometric average of the stock prices, then

$$\bar{X}_T = (r-q) + \frac{1}{T}\int_0^T X_s ds, \quad \hat{S}_T = e^{\bar{X}_T} = \exp\left((r-q) + \frac{1}{T}\int_0^T X_s ds\right). \tag{34}$$

For average strike we shall need the cumulant of integrated log-returns.

Corollary 2. *If (X, V) is an ASV model, then the law of $Y_t = \int_0^t X_s ds$ is described by*

$$\log E[e^{wY_t}] = \Phi(t, w) + wtX_0 + V_0\psi(t, w) \tag{35}$$

where

$$\dot{\Phi} = F(wt, \psi) \qquad \Phi(0) = 0 \tag{36}$$

$$\dot{\psi} = R(wt, \psi) \qquad \psi(0) = 0. \tag{37}$$

Proof. This follows immediately from Prop.2 with $u_1 = 0$, $u_2 = 0$, $u_3 = w$, $u_4 = 0$. □

Theorem 2. *Assume there exists $a > 1$ such that*

$$E[e^{a\bar{X}_T}] < \infty, \tag{38}$$

then the time-zero value of an average price Asian call option is given by

$$E[e^{-rT}(\hat{S}_T - K)_+] = \frac{e^{-rT}}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left(\frac{1}{K}\right)^u \frac{K}{u(u-1)} e^{\kappa(T,u)} du, \tag{39}$$

with the cumulant function $\kappa(T, u) = \log E[e^{u\bar{X}_T}]$. It is given by

$$\kappa(T, u) = u(r-q) + \phi(T, u) + uX_0 + \psi(T, u)V_0, \tag{40}$$

where

$$\dot{\phi} = F\left(\frac{ut}{T}, \psi\right) \qquad \phi(0) = 0 \tag{41}$$

$$\dot{\psi} = R\left(\frac{ut}{T}, \psi\right) \qquad \psi(0) = 0. \tag{42}$$

Proof. In order to evaluate the expectation (39) we first use the integral representation (132), which yields

$$(\hat{S}_T - K)_+ = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left(\frac{1}{K}\right)^u \frac{K}{u(u-1)} e^{u\bar{X}_t} du, \tag{43}$$

and then apply Fubini's Theorem, see also [HKK06] and [HS11]. □

Remark 3. *The integrability condition (38) guarantees the existence of the cumulant function $\kappa(T,u)$ at $\Re u = a$. It will imply some restrictions on the parameters of the concrete models studied in Section 6. By the results in [KRM13] it is equivalent to the existence of solutions of the Riccati equations* (41) *for the parameter value $u = a$. The proper set of parameters can be determined individually for each concrete model by studying the real singularities of the cumulant functions, see [Doe71, Satz 3.4.1, P.153f], though we are not going to give all details for all models in the example section below.*

5.2 Average strike

Theorem 3. *If there exists $b < 0$ such that*

$$E[e^{b\bar{X}_T}] < \infty, \tag{44}$$

then the time-zero value of an average strike Asian call option is given by

$$E[e^{-rT}(S_T - \hat{S}_T)_+] = \frac{e^{-qT}}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{1}{u(u-1)} e^{\kappa(T,u)} du, \tag{45}$$

where $\kappa(T,u) = \log E[e^{u\bar{X}_T + (1-u)X_T}]$. It is given by

$$\kappa(T,u) = u(r-q) + \phi(T,u) + V_0\psi(T,u) + X_0 \tag{46}$$

where

$$\dot{\phi} = F\left(\frac{ut}{T} + (1-u), \psi\right) \qquad \phi(0) = 0 \tag{47}$$

$$\dot{\psi} = R\left(\frac{ut}{T} + (1-u), \psi\right) \qquad \psi(0) = 0. \tag{48}$$

Proof. Using the change-of-numeraire technique with the density process (24) we obtain

$$E^0[(S_T - \hat{S}_T)_+] = e^{(r-q)T} E^1[(1 - e^{\bar{X}_T - X_T})_+]. \tag{49}$$

This is just the payoff of a put option on $e^{\bar{X}_T - X_T}$ with asset and strike both equal to 1.

The function κ is the cumulant function of $\bar{X}_T - X_T$, which can be obtained from the joint cumulant of Y_T and X_T in terms of the functions ϕ and ψ from Lemma 2.

Similar to the proof of Theorem 2, we can now apply the Laplace integral formula (133) provided in the appendix and Fubini's Theorem to obtain the result. □

Remark 4. *For the integrability condition (44) a remark similar to Remark 3 above applies.*

Proposition 3. *Average strike and price Riccati equations have the same structure provided the parameters are changed in the following way. $u \mapsto u + \frac{t}{T}(1-u)$.*

Remark 5. *The property just described in the proposition above is actually a particular case of a general result called the duality principles in option pricing. This basic property has been systematically investigated in a general semimartingale setting in [Pap07] and in [EPS08].*

6 Geometric Asian options for concrete affine stochastic volatility models

We now discuss some popular ASV models from the finance literature (for a very nice summary and many more examples, the interested reader is referred to [Kal06]). For a few relevant cases we will obtain an explicit solution of the corresponding Riccati equations. Let us recall, that for all models the asset price will be modeled by $S_t = e^{(r-q)t+X_t}$, where X denotes the discounted log-price.

In the following examples we shall continue to assume that the model parameters will verify the conditions in Proposition 1, and consequently e^{X_t} is a martingale.

6.1 Heston model

The Heston [Hes93] model describes the volatility dynamics by means of a CIR-type stochastic differential equation with mean reversion.

The evolution of the discounted log-returns under the risk-neutral measure is then given by

$$dX_t = \left(-\frac{1}{2}V_t\right)dt + \sqrt{V_t}dW_t^1, \tag{50}$$

$$dV_t = \lambda(\theta - V_t)dt + \zeta\sqrt{V_t}dW_t^2, \tag{51}$$

where λ, θ, and ζ are strictly positive parameters. Moreover, in (50) and (51) W^1 and W^2 are standard Wiener processes having constant correlation $\rho \in [-1, +1]$.

It can be shown that, if the following condition is satisfied:

$$\zeta^2 < 2\lambda\theta, \tag{52}$$

then the volatility process V remains strictly positive (see [Fel51]).

The affine characteristics are [KR08, KR11]

$$F(u,w) = \lambda\theta w, \quad R(u,w) = \frac{1}{2}(u^2 - u) + \frac{\zeta^2}{2}w^2 - \lambda w + uw\rho\zeta. \tag{53}$$

Average price

Combining (53) and (41) we obtain the Riccati equation for the average price

$$\dot{\phi} = \lambda\theta\psi, \quad \phi(0) = 0 \tag{54}$$

$$\dot{\psi} = \frac{\zeta^2}{2}\psi^2 - (\lambda - \rho\zeta ut/T)\psi + \frac{1}{2}ut/T(ut/T - 1), \quad \psi(0) = 0 \tag{55}$$

By using a standard substitution

$$\psi(t) = \frac{2}{\zeta^2}\frac{y'(t)}{y(t)} \tag{56}$$

the Riccati equation can be transformed into a linear differential equation of second order [Rei72, PZ03]

$$y'' + (\lambda - \rho\zeta ut/T)y' + \frac{\zeta^2}{4}ut/T(ut/T - 1) = 0. \tag{57}$$

The general solution of this equation can be written as a linear combination of two confluent hypergeometric functions of the first kind [Sla60] (most commonly denoted by ${}_1F_1(a,b,c)$)

$$y(t) = C_1 y_1(t) + C_2 y_2(t) \tag{58}$$

$$y_1(t) = AM\left(a_1, \frac{1}{2}, c\right) \tag{59}$$

$$y_2(t) = ABM\left(a_1 + \frac{1}{2}, \frac{3}{2}, c\right) \tag{60}$$

where $a1$, c and A,B are defined by the following expressions:

$$a_1 = \frac{1}{8}\frac{-\zeta^2 - 2\rho\zeta(\xi u/T - \lambda) - 2\lambda^2}{\zeta u/T\xi^{\frac{3}{2}}} + \frac{1}{4} \tag{61}$$

$$c = \frac{1}{2}\frac{((1 + tu/T\zeta)\xi - \lambda\rho)^2}{\zeta u/T\xi^{\frac{3}{2}}} \tag{62}$$

$$\xi = \rho^2 - 2 \tag{63}$$

$$A = \exp -\frac{1}{4}[t(2\lambda - \rho\zeta ut/T + \frac{\rho^2\zeta ut/T - 2\lambda\rho + 2\zeta(1 - ut/T)}{\xi^{\frac{1}{2}}}] \tag{64}$$

$$B = \xi ut/T\zeta - \xi + \lambda\rho \tag{65}$$

By taking into account the initial condition $\psi(0) = 0$ we obtain

$$\psi(t) = -\frac{2}{\zeta^2}\frac{y_2'(0)y_1'(t) - y_1'(0)y_2'(t)}{y_2'(0)y_1(t) - y_1'(0)y_2(t)}. \tag{66}$$

In view of the (56) we express also ϕ explicitly by hypergeometric functions, namely

$$\phi(t) = -\lambda\theta\frac{2}{\zeta^2}\ln\frac{y_2'(0)y_1(t) - y_1'(0)y_2(t)}{y_2'(0)y_1(0) - y_1'(0)y_2(0)} \tag{67}$$

Average strike

Combining (53) and (114) with $(u, w) \mapsto (-u, u/T)$ we obtain

$$\dot{\phi} = \lambda\theta\psi + q - r \qquad \phi(0) = 0 \tag{68}$$

$$\dot{\psi} = \frac{1}{2}u^2(t/T - 1)^2 + u(t/T - 1) + \frac{\zeta^2}{2}\psi^2 - \lambda\psi + \rho\zeta(u(t/T - 1) + 1)\psi. \qquad \psi(0) = 0 \tag{69}$$

The solution to these equations can be obtained in a similar way, providing the expressions for ϕ and ψ analogous to (66) and (67) with y_1, y_2 replaced by

$$\bar{y}_1(t) = M(\bar{a}_1, \frac{1}{2}, \bar{c}) \tag{70}$$

and

$$\bar{y}_2(t) = \bar{A}\bar{B}M(\bar{a}_1 + \frac{1}{2}, \tfrac{3}{2}, \bar{c}). \tag{71}$$

where $\bar{a}_1$, $\bar{c}$, $\bar{A}$ and $\bar{B}$ are now defined by:

$$\bar{a}_1 = \frac{1}{8}\frac{\rho\zeta(2\xi u + \lambda T) - (\lambda^2 + \frac{1}{4}\zeta^2)T}{\zeta u\xi^{\frac{3}{2}}} + \frac{1}{4} \tag{72}$$

$$\bar{c} = \frac{1}{2}\frac{[(\rho^2(u-1)+(\frac{1}{2}-u)]T+\zeta ut(1-\rho^2)+\lambda\rho T)^2}{\zeta uT\xi^{\frac{3}{2}}} \tag{73}$$

$$\bar{A} = \exp\frac{1}{2}[\frac{t}{T}(\rho\zeta((u-1)T-\frac{1}{2}ut)+\lambda T)ut+\frac{((\rho^2(u-1)-u+\frac{1}{2})-\frac{1}{2}\zeta u\frac{t}{T}(\rho^2-1))+\rho\lambda)}{\xi^{\frac{1}{2}}}] \tag{74}$$

$$\bar{B} = \{[(u-1)\rho^2-u+\frac{1}{2}]T+ut(1-\rho^2)\}\zeta+\lambda\rho T \tag{75}$$

Remark 6. *The pricing of Geometric Asian options in Heston's model has been investigated by Kim and Wee in [KW11]. They express the joint moment generating function of returns and integral average in terms of some series expansions. In fact, their series can be summed in closed form in terms of hypergeometric functions and agrees with our results above.*

6.2 The Bates model

In a model proposed by Bates [Bat96], a jump component is introduced in the previous dynamics for the log-returns by means of the compound Poisson process Z:

$$Z_t = \sum_{i=1}^{N_t} J_i, \tag{76}$$

where N is a standard Poisson process with intensity $\nu > 0$ and (J_i), $i = 1, 2, 3, \ldots$, are independent random variables, all having a normal distribution with mean γ and standard deviation δ. In such a case the Lévy measure of Z is given by:

$$U(dx) = \frac{\nu}{\delta\sqrt{2\pi}}\exp\left[-\frac{(x-\gamma)^2}{2\delta^2}\right], \tag{77}$$

and the cumulant function of Z takes the form:

$$\kappa(z) = \nu(e^{\gamma z+\delta^2 z^2/2}-1). \tag{78}$$

The dynamics of discounted log-returns under the risk-neutral measure is then given by:

$$dX_t = (-\kappa(1)-\frac{1}{2}V_t)dt+\sqrt{V_t}dW_t^1+dZ_t, \tag{79}$$

and the dynamics of the volatility is the same as that proposed by the Heston model, namely

$$dV_t = \lambda(\theta-V_t)dt+\zeta\sqrt{V_t}dW_t^2. \tag{80}$$

The affine characteristics are

$$F(u,w) = \lambda\theta w+\kappa(u)-u\kappa(1), \quad R(u,w) = \frac{1}{2}(u^2-u)+\frac{\zeta^2}{2}u_2^2-\lambda w+uw\rho\zeta. \tag{81}$$

If $\nu \to 0$, then we obtain the Heston stochastic volatility model [Hes93]. If $\zeta \to 0$ and $V_0 = \theta$ then $V_t = \theta$ we obtain the Merton jump-diffusion model [Mer76]. Consequently we might consider the Bates model as an extension of a Merton model to the case of stochastic volatility, or as an extension of the Heston stochastic volatility model to the case of jumps in the asset prices.

In the Bates model the Riccati equations for the average price are

$$\dot{\phi} = \lambda\theta\psi + \kappa(ut) - ut\kappa(1), \quad \phi(0) = 0 \tag{82}$$

$$\dot{\psi} = \frac{\zeta^2}{2}\psi^2 - (\lambda - \rho\zeta ut)\psi + \frac{1}{2}ut(ut-1), \quad \psi(0) = 0. \tag{83}$$

We observe that the equation for ψ is exactly the same as in the Heston model and ϕ equals the corresponding quantity from the Heston model plus an integral of the cumulant of the jumps (quite easy to compute):

$$\phi(t) = \phi_H(t) + \int_0^t \kappa(us)ds - ut\kappa(1), \tag{84}$$

where ϕ_H is given above in (67). This is due to the fact, that the jumps are independent of the continuous part.

For the average strike we obtain the same ψ as in the Heston model, while the ϕ is provided by the following expression (also easy to compute):

$$\phi(t) = \phi_H(t) + \int_0^t \kappa\left(u\left(\frac{t}{T} - 1\right)u + 1\right)ds - \frac{ut}{T}\kappa(1) + (1-u)\kappa(1), \tag{85}$$

where ϕ_H is given above, using (70)and (71).

6.3 The Turbo-Bates model

In [Bat00] Bates introduced a refinement of the previous model with state-dependent jump intensity. Following [KR11, Sec.6.2] we will consider a simplified version with only one variance factor. The risk-neutral dynamics for log-returns are given by

$$dX_t = \left(-\nu_0\kappa(1) - \left(\frac{1}{2} + \nu_1\kappa(1)\right)V_t\right)dt + \sqrt{V_t}dW_t^1 + \int_D x\tilde{N}(V_t, dt, dx) \tag{86}$$

$$dV_t = -\lambda\left(V_t - \theta\right)dt + \zeta\sqrt{V_t}dW_t^2$$

where $\lambda, \theta, \zeta > 0$ as before and the Brownian motion W^1, W^2 are correlated with correlation coefficient ρ. The jump component is given by $\tilde{N}(V_t, dt, dx) = N(V_t, dt, dx) - \mu(V_t, dt, dx)$, where $N(V_t, dt, dx)$ is a Poisson random measure and its predictable compensator $\mu(V_t, dt, dx) = (\nu_0 + \nu_1 V_t)F(dx)dt$, and F is some fixed jump size distribution.

The affine characteristics are

$$F(u,w) = \nu_0\kappa(u) - u\nu_0\kappa(1)0 + \lambda\theta w, \quad R(u,w) = \frac{1}{2}(u^2-u) + \frac{\zeta^2}{2}w^2 - \lambda w + \rho\zeta uw + \nu_1\kappa(u) - u\nu_1\kappa(1) \tag{87}$$

where $\kappa(u)$ is the cumulant generating function of F.

The Riccati equations for the average price are

$$\dot{\phi} = \lambda\theta\psi + \nu_0\kappa(ut/T) - ut/T\nu_0\kappa(1), \quad \phi(0) = 0 \tag{88}$$

$$\dot{\psi} = \frac{\zeta^2}{2}\psi^2 - (\lambda - \rho\zeta ut/T)\psi + \frac{1}{2}ut/T(ut/T - 1) + \nu_1\kappa(ut/T) - ut/T\nu_1\kappa(1), \quad \psi(0) = 0 \tag{89}$$

and for the average strike

$$\dot{\phi} = \alpha\theta\psi + \nu_0\kappa(u(t/T-1)+1) - \left(\frac{ut}{T} + (1-u)\right)\nu_0\kappa(1), \quad \phi(0) = 0 \tag{90}$$

$$\dot{\psi} = \frac{1}{2}u^2(t/T-1)^2 + u(t/T-1) + \frac{\zeta^2}{2}\psi^2 - \beta\psi + \rho\zeta(u(t/T-1)+1)\psi + \tag{91}$$

$$+ \lambda_1\kappa((u(t/T-1)+1)) - \left(\frac{ut}{T} + (1-u)\right)\nu_1\kappa(1), \tag{92}$$

$$\psi(0) = 0. \tag{93}$$

6.4 Barndorff-Nielsen-Shephard model

The BNS model has been introduced by Ole Barndorff-Nielsen and Neil Shephard. [BNS01] , [BNNS02], [NV03], [HS09]. The model is constructed from a subordinator, called background driving Lévy process (BDLP), with cumulant generating function

$$\kappa(\theta) = \log E[e^{\theta Z(1)}], \tag{94}$$

which exists for $\Re(\theta) < \ell$ with some real number $\ell > 0$. The instantaneous variance process $(V(t), t \geq 0)$ is described by the following stochastic differential equation of Ornstein-Uhlenbeck type,

$$dV(t) = -\lambda V(t-)dt + dZ_\lambda(t), \tag{95}$$

with $V_0 > 0$ and $\lambda > 0$ given real numbers. The logarithmic return process $(X(t), t \geq 0)$ is given by:

$$dX(t) = (-\kappa(\rho) - \frac{1}{2}V(t-))dt + \sqrt{V(t-)}dW(t) + \rho dZ_\lambda(t), \quad X(0) = 0, \tag{96}$$

with parameters $\mu \in$, $\beta \in$, $\rho \leq 0$. The affine characteristics are

$$F(u,w) = \lambda k(w + \rho u) - u\lambda k(\rho), \quad R(u,w) = \frac{1}{2}(u^2 - u) - \lambda w. \tag{97}$$

Riccati equations for average price are

$$\dot{\phi} = \lambda k(\psi + \rho ut/T) - ut/T\lambda k(\rho) \qquad \phi(0) = 0 \tag{98}$$

$$\dot{\psi} = \frac{1}{2}(u^2t^2/T^2 - ut/T) - \lambda\psi \qquad \psi(0) = 0. \tag{99}$$

We remark the equation for ψ is linear and can be solved explicitly, giving

$$\psi(t) = \frac{u^2}{2T^2}f_2(t) - \frac{u}{tT}f_1(t). \tag{100}$$

with

$$f_0(t) = \frac{1-e^{-\lambda t}}{\lambda}, \qquad f_1(t) = \frac{t}{\lambda} - \frac{1-e^{-\lambda t}}{\lambda^2}, \qquad f_2(t) = \frac{t^2}{\lambda} - \frac{2t}{\lambda^2} + \frac{2(1-e^{-\lambda t})}{\lambda^3}. \tag{101}$$

The equation for ϕ yields an integral

$$\phi(t) = \int_0^t \lambda k\left(\psi(s) + \rho us/T\right) - \frac{t^2}{2}\frac{u}{T}\lambda k(\rho) \tag{102}$$

Riccati equations for average strike are

$$\dot{\phi} = \lambda k(\psi + \rho(u(t/T - 1) + 1)) \qquad \phi(0) = 0 \tag{103}$$

$$\dot{\psi} = \frac{1}{2}u^2(t/T - 1)^2 + u(t/T - 1) - \lambda\psi \qquad \psi(0) = 0. \tag{104}$$

The solution is quite analogous, now with

$$\psi(t) = \frac{u^2}{2T^2} f_2(T) + \frac{u}{T}\left(\frac{1}{2} - u\right) f_1(T) + \left(\frac{u^2}{2} - \frac{u}{2}\right) f_0(T). \tag{105}$$

and the integral

$$\phi(t) = \int_0^T \lambda k \left(\psi(s) + \rho\left(\left(\frac{s}{T} - 1\right) u + 1\right)\right) ds - \left(\left(1 - T\right) u + T\right) \lambda k(\rho). \tag{106}$$

The results agree[1] with those from [HS11], which were obtained by a different technique without employing the general affine framework and Riccati equations.

6.5 OU time-changed Lévy processes

Time-changed Levy processes have been introduced by P. Carr, H. Geman, D. Madan and M. Yor [CGMY03] in order to improve Levy models performances in describing asset price dynamics. We shall concentrate our attention on time changes based on processes satisfying a stochastic differential equation of an Ornstein-Uhlenbeck or a square-root (CIR) type. Let L be a Lévy process with cumulant function

$$\theta(u) = \log E[e^{uL(1)}]. \tag{107}$$

Then we define

$$X_t = L(\Gamma(t)), \tag{108}$$

where $\Gamma(t)$ is a non-negative increasing process independent of L. Here we would like to use a very popular time change, namely an integrated Ornstein-Uhlenbeck (OU) type process.

Definition 1 (OU time-change)**.** *The OU time-change model is given as*

$$\Gamma(t) = \int_0^t V(s)ds, \tag{109}$$

where V is now given as solution of the SDE

$$dV(t) = -\lambda V(t)dt + dU(t), \tag{110}$$

with U being a pure jump subordinator with cumulant function $\kappa(u)$.

The affine characteristics are

$$F(u, w) = \lambda\kappa(w), \quad R(u, w) = -\lambda w + \theta(u). \tag{111}$$

For the Riccati equations we get from (41) and (111)

$$\dot{\phi} = \lambda\kappa(\psi) \qquad \phi(0) = 0 \tag{112}$$

$$\dot{\psi} = -\lambda\psi + \theta(ut) \qquad \psi(0) = 0. \tag{113}$$

[1] Actually term $-u/2$ is missing in [HS11, (47)] and should be included there.

The Riccati equations for average strike are

$$\dot{\phi} = \lambda\kappa(\psi) + q - r \qquad \phi(0) = 0 \tag{114}$$

$$\dot{\psi} = -\lambda\psi + \theta(u(t/T - 1) + 1) \qquad \psi(0) = 0 \tag{115}$$

Let us consider a concrete example of a time-changed Lévy given by a Kou double exponential Lèvy process with time change implied by an integrated OU process. By recalling that the cumulant for the double exponential has the following expression:

$$\kappa(u) = \nu u \left[\frac{p}{\alpha_+ - u} - \frac{1-p}{\alpha_- + u} \right], \tag{116}$$

where ν is the intensity of the jump process, α_-, α_+ describe the exponential tails, the Riccati equations for average price have the following explicit solution:

$$\psi(t) = -\, e^{-\lambda t} \frac{\nu}{\lambda u} \left\{ [p \operatorname{Ei}(1, -\frac{\lambda\alpha_-}{u}) e^{(\frac{\lambda\alpha_-}{u})} \lambda\alpha_- \operatorname{Ei}(1, \frac{\lambda\alpha_+}{u}) e^{(\frac{\lambda\alpha_+}{u})} \lambda\alpha_+ + u + p\lambda\alpha_+ \operatorname{Ei}(1, \frac{\lambda\alpha_+}{u}) e^{(\frac{\lambda\alpha_+}{u})} \right\} \tag{117}$$

$$-\, e^{-\lambda t} \frac{\nu}{\lambda u} \left\{ [p \operatorname{Ei}(1, -\frac{\lambda(\alpha_- + ut)}{u}) e^{(\frac{\lambda(\alpha_- + ut)}{u})} \lambda\alpha_- \operatorname{Ei}(1, \frac{\lambda(\alpha_+ - ut)}{u}) e^{(\frac{\lambda(\alpha_+ - ut)}{u})} \lambda\alpha_+ \right. \tag{118}$$

$$\left. + u + p\lambda\alpha_+ \operatorname{Ei}(1, \frac{\lambda(\alpha_+ - ut)}{u}) e^{(\frac{\lambda(\alpha_+ - ut)}{u})} \right\} \tag{119}$$

While the average strike Riccati equations have the following solution:

$$\psi(t) = e^{-\lambda t} \frac{\nu}{u} [p\lambda\alpha_- T \operatorname{Ei}(1, -\frac{\lambda\alpha_- T - \lambda u T - \lambda u(T-t)}{u}) e^{(\frac{\lambda\alpha_- T - \lambda u T - \lambda u(T-t)}{u})} + u] \tag{120}$$

$$-e^{-\lambda t} \frac{\nu}{u} [p\lambda\alpha_+ T \operatorname{Ei}(1, \frac{\lambda\alpha_+ T - \lambda u T - \lambda u(T-t)}{u}) e^{(\frac{\lambda\alpha_+ T - \lambda u T - \lambda u(T-t)}{u})}] +$$

$$-e^{-\lambda t} \frac{\nu}{u} [-p\lambda\alpha_+ T \operatorname{Ei}(1, -\frac{\lambda(\alpha_- T - \lambda u T - \lambda u(T-t))}{u}) e^{(\frac{\lambda(\alpha_- T - \lambda u T - \lambda u(T-t))}{u})}]$$

6.6 CIR time-changed Lévy processes

Another time change which has been proposed in [CGMY03] for a Levy process in order to improve its performances in describing logreturns statistical behavior, is that driven by an integrated CIR process, i.e., a process satisfying the following SDE:

$$dV_t = -\lambda\,(V_t - \theta)\,dt + \eta\sqrt{V_t} dW_t. \tag{121}$$

The time-change and the returns process will be given by (109) and (108) as above.

The affine characteristics are

$$F(u, w) = \lambda\theta w, \quad R(u, w) = \frac{\eta^2}{2} w^2 - \lambda w + \kappa(u). \tag{122}$$

where $\kappa(u)$ is the cumulant generating function of the Lévy process.

From (41) and (122) we obtain the Riccati equations for average price

$$\dot{\phi} = \lambda\theta\psi \qquad \phi(0) = 0 \tag{123}$$

$$\dot{\psi} = \frac{\eta^2}{2}\psi^2 + -\lambda\psi + \kappa(ut) \qquad \psi(0) = 0. \tag{124}$$

For average strike

$$\dot{\phi} = \lambda\theta\psi + q - r \qquad \phi(0) = 0 \tag{125}$$

$$\dot{\psi} = \frac{\eta^2}{2}\psi^2 + -\lambda\psi + \kappa(u(t/T - 1) + 1) \qquad \psi(0) = 0 \tag{126}$$

Let us consider a concrete example of a time-changed Lévy given by a Kou double exponential Lèvy process with time change implied by an integrated CIR process. By recalling that the cumulant for the double exponential has the following expression:

$$\kappa(u) = \nu u \left[\frac{p}{\alpha_+ - u} - \frac{1-p}{\alpha_- + u}\right], \tag{127}$$

where ν is the intensity of the jump process, α_-, α_+ describe the exponential tails, the Riccati equation for ψ becomes:

$$\dot{\psi} = \frac{\eta^2}{2} - \nu\dot{\psi}\psi^2 + \lambda u \left[\frac{p}{\nu_- - u} - \frac{1-p}{\nu_+ + u}\right], \psi_w(0) = 0. \tag{128}$$

For a symmetric jump distribution, i.e., $p = 1/2$ and $\alpha_+ = \alpha_-$ we can provide an explicit solution in terms of *Heun's* confluent hypergeometric function C, see [Ron95, SK10]:

$$\psi(t) = -\frac{2}{\eta^2}\frac{y_2'(0)y_1'(t) - y_1'(0)y_2'(t)}{y_2'(0)y_1(t) - y_1'(0)y_2(t)}. \tag{129}$$

$$y_1 = \exp(-\frac{\lambda t}{2})(\alpha_+^2 - u^2t^2)C(0, -\frac{1}{2}, 1, \frac{-\lambda^2\alpha_+^2 + 2et^2\nu\alpha_+}{16u^2}, \frac{8u^2 + \lambda^2\alpha_+^2}{16u^2}, \frac{u^2t^2}{\alpha_+^2}) \tag{130}$$

$$y_2 = \exp(-\frac{\lambda t}{2})(\alpha_+^2 - u^2t^2)C(0, +\frac{1}{2}, 1, \frac{-\lambda^2\alpha_+^2 + 2et^2\nu\alpha_+}{16u^2}, \frac{8u^2 + \lambda^2\alpha_+^2}{16u^2}, \frac{u^2t^2}{\alpha_+^2}) \tag{131}$$

7 Concluding Remarks

In this paper we have just provided a framework for Geometric Asian options valuation and we have shown that this valuation problem, through the general affine process approach, can be reduced to solving some generalized Riccati equations and that in many relevant cases these equations admit close-form solutions. The final step of the present valuation procedure requires the numerical inversion of a Laplace transform. This computation, which has become quite standard in option pricing, nevertheless requires some care especially when complicated special functions, like those considered insofar, are involved. The research of a fast and accurate algorithm providing such inversion will be the subject of our future investigation together with an extensive comparison of the numerical methods available for Geometric Asian option pricing in affine stochastic volatility models. As we mentioned in Section 5, proper integrability conditions must be verified in order to apply our general pricing results: the existence of all the involved cumulant functions must be assured; this can be investigated through the analysis of the singularities of the special functions introduced. This subject, together with a systematic numerical illustration of the present results will be the subject of our future investigation and will be collected in a separate paper.

A Laplace formulae

Lemma 3. *Suppose we are given real numbers $S_0 > 0$, $K > 0$, and $a > 1$, $0 < b < 1$, $c < 0$. Then we have for all $x \in \mathbb{R}$ the formulas*

$$(e^x - K)_+ = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \left(\frac{1}{K}\right)^u \frac{K}{u(u-1)} e^{ux} du, \tag{132}$$

$$(K - e^x)_+ = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\frac{1}{K}\right)^u \frac{K}{u(u-1)} e^{ux} du, \tag{133}$$

and

$$(e^x - K)_+ - e^x = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \left(\frac{1}{K}\right)^u \frac{K}{u(u-1)} e^{ux} du. \tag{134}$$

Proof. Let $f(x) = (e^x - K)_+$. An elementary calculation provides the (bilateral) Laplace transform of f, namely

$$\int_{-\infty}^{+\infty} f(x) e^{-ux} dx = \left(\frac{1}{K}\right)^u \frac{K}{u(u-1)} \tag{135}$$

for $\Re u > 1$. Now f is continuous and has locally bounded variation, which are sufficient conditions to guarantee that the Laplace inversion integral (with Bromwhich contour) yields the original function, that is (132). See [Doe71, Satz 4.4.1, P.210]. The proof for (133) and (134) is similar.
□

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