Reducing model risk via positive and negative dependence assumptions

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Abstract

We give analytical bounds on the Value-at-Risk and on convex risk measures for a portfolio of random variables with fixed marginal distributions under an additional positive dependence structure. We show that assuming positive dependence information in our model leads to reduced dependence uncertainty spreads compared to the case where only marginals information is known. In more detail, we show that in our model the assumption of a positive dependence structure improves the best-possible lower estimate of the risk while it does not help to increase the lower risk bounds in an essential way. As a result we find that additional assumptions on the dependence structure may result in essentially improved risk bounds.

Keywords: Model risk, Dependence uncertainty, Positive dependence, Value-at-Risk, Convex risk measures


1. Preliminaries and motivation

The problem of assessing the model risk associated with the risk measurement of a high dimensional portfolio has recently gathered a lot of interest in the actuarial and financial literature. To set a mathematical framework, we assume that a financial institution holds a $d$-dimensional risk portfolio over a fixed time period. This risk portfolio is represented by a random vector $X=(X_1,\ldots,X_d)$ on a standard atomless probability space $(\Omega,\mathcal{F},P)$. The total loss exposure associated with $X$ is given by the sum

\[ X_d^+ = X_1 + \cdots + X_d. \]

Using a risk measure $\rho$, the aggregate random position $X_d^+$ is mapped into the real value $\rho(X_d^+)$, to be interpreted as the regulatory capital to be reserved in order to be able to safely hold $X$. In this paper, we mainly deal with the case where $\rho$ is a convex risk measure or the case where $\rho$ is the Value-at-Risk (VaR). The evaluation of $\rho(X_d^+)$ is mainly a numerical issue once the joint distribution of $X$ has been chosen or statistically evaluated. Estimating a multivariate distribution is a challenging task which is usually performed in two steps: first, $d$ individual models $F_j$ for the marginal loss exposures $X_j$ are independently developed. Then, the marginal distributions are merged into a joint distribution using a dependence structure.

In fact, banks/insurance companies typically have better methods/more data for estimating a one-dimensional distribution for each risk type $X_j$ than they have to estimate the overall dependence structure of $X$. It is therefore reasonable to assume that the marginal distributions $F_1,\ldots,F_d$ are known, while $F_X$, the joint distribution of $X$, varies in $\mathcal{F}_d(F_1,\ldots,F_d)$, the so-called Fréchet class of all possible joint distributions having the fixed marginal models $F_1,\ldots,F_d$. The choice of a single distribution in $\mathcal{F}_d(F_1,\ldots,F_d)$ can lead to the miscalculation of the reserve $\rho(X_d^+)$. The implied model risk is referred to as dependence uncertainty.

A natural way to measure dependence uncertainty and, in more generality, model risk consists in finding the minimum and maximum possible values of the risk measure $\rho$ evaluated over the class of candidate models; this is the approach taken in \cite{Cont1}. In our framework, we define the smallest and biggest capitals to be held coherently with the given marginal distributions as

\[ \underline{\rho}(X_d^+) = \inf \left\{ \rho(X_d^+); F_X \in \mathcal{F}_d(F_1,\ldots,F_d) \right\}, \]

and

\[ \overline{\rho}(X_d^+) = \sup \left\{ \rho(X_d^+); F_X \in \mathcal{F}_d(F_1,\ldots,F_d) \right\}. \]

For any risk portfolio $(X_1,\ldots,X_d)$ having marginal distributions $F_1,\ldots,F_d$, it obviously holds that

\[ \underline{\rho}(X_d^+) \leq \rho(X_d^+) \leq \overline{\rho}(X_d^+). \]

The difference $\overline{\rho}(X_d^+) - \underline{\rho}(X_d^+)$ is called Dependence Uncertainty spread (DU-spread) for $\rho$ and is used to measure model uncertainty on the final capital reserve; see \cite{Embrechts1} for this terminology.

Computation of DU-spreads has been treated in the recent literature. The analytical computation of best- and worst-possible bounds on VaR can be performed only under some specific assumptions on the marginal distributions; see the survey pa-
per Embrechts et al. (2014a) for the state-of-the-art and a history of the problem. The analytical computation of worst-possible bounds on Expected Shortfall (ES) is in general straightforward, while for the best-case ES partial analytical results can be found in Wang and Wang (2011) and Bernard et al. (2014).

For several classes of risk measures (including convex and distortion risk measures) Wang et al. (2014) provide a systematic way to compute the worst (and best) possible bounds across any homogeneous portfolio.

The numerical computation of DU-spreads of VaR and ES for arbitrary portfolios can be performed using the Rearrangement Algorithm described in Embrechts et al. (2013) (for the case of VaR) and in Puccetti (2013) (for ES) for dimensions $d$ in the several hundreds or possibly thousands. Even if DU-spreads of VaR and ES are numerically available for practically any joint portfolio of risks, their relevance in actuarial practice has been recently questioned since they can be considerably large; see Aas and Puccetti (2014) for a real case study.

Therefore, in the recent literature many techniques to tighten DU-spreads were introduced. One possibility is to add extra (statistical) information on the topology of the marginal distributions. For instance, in Embrechts et al. (2013) Section 4 it is shown that having higher order (typically two-dimensional) marginals information on the joint portfolio leads to strongly improved bounds. The DU-spread of the VaR can be similarly reduced by specifying the copula on some subset of its domain (see Bernard et al. 2013a) or putting a variance constraint on the total position (see Bernard et al. 2013b).

In this paper we follow the idea to improve lower and upper risk bounds by introducing positive, respectively negative, dependence restrictions. Some results in this direction have been considered in the literature as in Williamson and Downs (1990), Denuit et al. (1999), Embrechts et al. (2003), Rüschendorf (2005), Embrechts and Puccetti (2006), Puccetti and Rüschendorf (2012).

In particular, we show that positive dependence restrictions do not help to improve upper risk bounds essentially. They however allow to increase the lower risk bounds and therefore to reduce the model risk faced by an institution. Positive dependence information is introduced in Section 2 by the notions of orthant orders and weakly conditional increasing in sequence order. These orders are particularly capable to capture the concept of stronger dependence in the comparison of portfolios with fixed marginal distributions. In Section 3 we introduce a class of models which are bounded below in some stochastic ordering by a random vector whose marginals are decomposed in several independent subgroups with comonotonic dependence within the subgroups. This assumption allows to model a relevant variety of positive dependence restrictions.

We provide analytical upper and lower bounds on the VaR of the joint portfolio which are easily computable and are compared with the corresponding unconstrained bounds obtained without positive dependence assumptions. In Section 4 we deal with the case of law-invariant, convex risk measures, where we draw similar conclusions. While assuming a positive dependence structure typically improves the best-possible lower bound of a risk measure, it generally leaves unchanged the worst-possible upper risk bound. Finally, in Section 5 we discuss how negative dependence assumptions moderate also worst-case scenarios. We give a variety of applications of interest in quantitative risk management that can be easily adapted and closed formulas to be used in the risk management of real portfolios.

2. Dependence orders between risk vectors

In quantitative risk management, the components of a risk portfolio often have some positive dependence structure. A simple way to describe positive dependence is by using suitable stochastic orders between random vectors. In this section, we recall some natural positive dependence orders needed in the remainder of the paper. For more details on these dependence notions we refer to Chapter 6 in Rüschendorf (2013).

For a random vector $X = (X_1, \ldots, X_d)$ in $\mathbb{R}^d$ we indicate with $F_X$ its joint distribution function and with $\bar{F}_X$ its survival function.

Formally, for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we denote

$$F_X(x) = P(X_1 \leq x_1, \ldots, X_d \leq x_d),$$

$$\bar{F}_X(x) = P(X_1 > x_1, \ldots, X_d > x_d).$$

For two random vectors $X$ and $Y$ in $\mathbb{R}^d$, we define

- the upper orthant order $Y \leq_{uo} X$, if $\bar{F}_Y(x) \leq \bar{F}_X(x)$ for all $x \in \mathbb{R}^d$;
- the lower orthant order $Y \leq_{lo} X$, if $F_Y(x) \leq F_X(x)$ for all $x \in \mathbb{R}^d$;
- the concordance order $Y \leq_{co} X$, if both $Y \leq_{uo} X$ and $Y \leq_{lo} X$ hold;
- the weakly conditional increasing in sequence order $Y \leq_{wcs} X$, if, for all $x \in \mathbb{R}$, all $i$ with $1 \leq i \leq d-1$, and all componentwise increasing functions $f : \mathbb{R}^{d-i} \to \mathbb{R}$, we have

$$\text{Cov}(1(Y_i > x), f(Y_{i+1}, \ldots, Y_d)) \leq \text{Cov}(1(X_i > x), f(X_{i+1}, \ldots, X_d)).$$  \hspace{1cm} (2.1)

A random vector $Y$ is smaller than $X$ in the upper (lower) orthant order if the probabilities for upper (lower) orthants are ordered, i.e. the probability that all components jointly assume large (small) values is lower for $Y$ than for $X$. In the case $Y$ has independent components with the same marginal distributions as $X$, condition (2.1) is equivalent to the condition that

$$F_{X_{i+1}|X_i > x} \geq_{st} F_{X_{i+1}}, \text{ for all } i \leq d - 1, x \in \mathbb{R},$$  \hspace{1cm} (2.2)

where $X_{i+1} = (X_{i+1}, \ldots, X_d)$. The above condition means that the conditional distribution of $X_{i+1}$ given $X_i > x$ is stochastically larger than the distribution of $X_{i+1}$, for all real thresholds $x$. In this case condition (2.1) postulates that $X$ is weakly associated in sequence, i.e. WAS. This indicates some form of positive dependence of $X$ and some consequences are described in Proposition 2.1 below. A slightly weaker dependence assumption than $Y \leq_{wcs} X$ is the supermodular order $Y \leq_{sm} X$ for random vectors having the same marginal distributions, $Y \leq_{wcs} X$.\hspace{1cm}
implies $Y \leq_{\text{sm}} X$; see Theorem 6.22 in [Rüschendorf 2013]. However, if compared to $\leq_{\text{sm}}$, the stochastic order $\leq_{\text{wcs}}$ is more easily interpretable and can be more easily checked in several functional models; see [Rüschendorf 2004].

It is well known that in dimension $d = 2$ and assuming identical marginals for the two vectors $X$ and $Y$ the four orders defined above are equivalent, i.e.,

$$Y \leq_{\text{wcs}} X \iff Y \leq_{\text{co}} X \iff Y \leq_{\text{lo}} (\leq_{\text{uo}}) X,$$

The four orders are however different when $d \geq 3$, where we have that

$$Y \leq_{\text{wcs}} X \Rightarrow Y \leq_{\text{co}} X \Rightarrow Y \leq_{\text{lo}} (\leq_{\text{uo}}) X,$$

but not vice versa. The orders $\leq_{\text{sm}}, \leq_{\text{lo}}, \leq_{\text{co}}, \leq_{\text{wcs}}$ all imply a stronger positive dependence when vectors with the same marginal distributions are to be compared. Here stronger positive dependence implies having bigger pairwise correlation or bigger rank correlation, as the following proposition shows; see Section 5.3 in [Denuit et al. 2005] and Remark 6.3 in [Rüschendorf 2013].

**Proposition 2.1.** Let $X$ and $Y$ be two random vectors having joint distributions $F_X, F_Y \in \mathcal{F}_d(F_1, \ldots, F_d)$ such that

$$Y \leq X,$$

where $\leq$ is one of the orders $\leq_{\text{sm}}, \leq_{\text{lo}}, \leq_{\text{co}}, \leq_{\text{wcs}}$. Then:

$$\text{Cov}(Y_i, Y_j) \leq \text{Cov}(X_i, X_j),$$

$$\rho_S(Y_i, Y_j) \leq \rho_S(X_i, X_j),$$

$$\tau(Y_i, Y_j) \leq \tau(X_i, X_j),$$

where $\rho_S$ is Spearman’s and $\tau$ is Kendall’s rank correlation coefficient.

In the remainder of this section we use the assumptions $Y \leq_{\text{lo}} X$ and $Y \leq_{\text{sm}} X$, while in Section 2 we assume the stronger condition $Y \leq_{\text{wcs}} X$. Orthant orders lead to bounds for the distribution function, resp. the survival function of an aggregate position. For a $d$-dimensional distribution function $G$, we define the generalized $G$-supremal convolution as

$$\sqrt{G(s)} = \sup_{x \in \mathcal{G}(s)} G(x),$$

where $\mathcal{G}(s) = \{(x_1, \ldots, x_d) \in \mathbb{R}^d; \sum_{i=1}^d x_i = s\}$.

The following proposition on improved bounds for the distribution function of the joint portfolio based on dependence information has been given in [Denuit et al. 1999]; see also [Puccetti and Rüschendorf 2012] for a detailed proof. It will be used in Section 3 to obtain analytical bounds for the VaR of a joint portfolio under positive dependence.

**Proposition 2.2.** Let $X$ and $Y$ be two random vectors in $\mathbb{R}^d$.

1. If $Y \leq_{\text{lo}} X$, then

$$P(X_i^s < s) \geq \sqrt{F_Y(s)}; \quad (2.3)$$

2. If $Y \leq_{\text{uo}} X$, then

$$P(X_i^s < s) \leq 1 - \sqrt{F_Y(s)} \quad (2.4)$$

The two bounds in (2.3) and (2.4) are sharp when $d \leq 2$.

3. Reducing the dependence spread of Value-at-Risk

In this section we introduce positive dependence conditions on the portfolio vector $X$ which allow to describe analytically the increase of the lower VaR bound as well as the decrease of the upper VaR bound from marginals information only. It will turn out that positive orthant dependence alone does not allow a reduction of the dependence spread of VaR. For this reason, in the following we introduce and study a more specific class of positive dependence models. Recall that the Value-at-Risk (VaR) of a loss random variable $X$, computed at a probability level $\alpha \in (0, 1)$, is defined as

$$\text{VaR}_\alpha(X) := F_X^{-1}(\alpha) = \inf\{x \in \mathbb{R}: F_X(x) \geq \alpha\},$$

where $F_X(x) = P(X \leq x)$ is the distribution function of $X$.

Consider a decomposition $\{1, \ldots, d\} = \bigcup_{k=1}^d I_k$ of $\{1, \ldots, d\}$ into $k$ disjoint subsets $I_j$ with cardinality $n_j = |I_j|$, $1 \leq j \leq k$, and let $Y$ denote a random vector with distribution function

$$F_Y(x_1, \ldots, x_d) = \prod_{j=1}^k \min\{G_j(x_i)\}, \quad (3.1)$$

where $G_1, \ldots, G_k$ are univariate distribution functions. By definition the components of $Y$ within the subgroups $I_j$ are homogeneous with distribution $G_j$ and comonotonic, while the different subgroups are independent. Throughout this section, our positive dependence restriction on $X$ is formulated by the condition that

$$Y \leq X, \quad (3.2)$$

where $\leq$ is one of the orders $\leq_{\text{sm}}$ or $\leq_{\text{lo}}$ as described in Section 2.

In general, condition (3.2) does not imply that the marginals of $Y$ are the same as the marginals of $X$. However, in applications we typically have the situation that $Y$ and $X$ have the same marginal distributions, that is

$$F_i = G_j, \text{ if } i \in I_j. \quad (3.3)$$

If $Y$ and $X$ have the same marginals and in the special case of $k = d$ groups, condition (3.2) postulates that $X$ is positive upper (resp. lower) dependent, i.e. PUOD (resp. PLOD). On the other extreme for $k = 1$ conditions (3.2) and (3.3) imply the (too) strong assumption that both $Y$ and $X$ are comonotonic vectors. The formulation in (3.2) however allows to include a large set of intermediate cases which may be quite relevant for applications in insurance.

Condition (3.1) postulates that the vector $Y$ has $k$ independent subgroups where the variables within the subgroups are comonotonic and homogeneous. In actuarial practice, very often some subgroups of the marginal risks can be assumed to
be independent. This is the case where for instance inhomogeneous sources of risks are gathered in a top-level portfolio where e.g. the marginal risks represent loss types aggregated over different subsidiary companies. As a representative example one might think that some risk categories such as catastrophic risk, operational risk and reinsurance risk can be assumed to be independent from market-driven losses deriving from market, credit, ownership risk.

Decomposition of a risk portfolio into independent subgroups is a very natural assumption that can be statistically checked and which does not imply the use/choice of an advanced model for the dependence structure (often leading to large statistical uncertainty). Once the risk portfolio is decomposed into subgroups, a popular assumption in actuarial practice would be to put some positive correlation structure within the subgroups.

The assumption of comonotonicity within the subgroups of the vector $Y$ may be too strong for several applications in insurance but it may serve to define worst case situations if one uses a convex risk measure to quantify the risk of the portfolio held. This is studied later in Section 4 when deriving risk bounds for convex risk measures. When one uses Value-at-Risk, the assumption of comonotonicity within the subgroups cannot be regarded as conservative from a mathematical viewpoint (VaR is well known not to be a coherent risk measure) but serves to rule out the worst-case VaR distributions attained by negative dependence as described in Wang and Wang (2011), Embrechts et al. (2013) and Aas and Puccetti (2014). However, it turns out that the upper bound on the VaR of the aggregate position will not be essentially reduced by positive dependence assumptions especially when the portfolio consists of a relatively large number of random variables. We discuss this issue further in Section 5.

The specific form of comonotonicity within the subgroups allows us to derive explicit analytical formulas for VaR bounds. We are now ready to give the main result of this section, where we state lower and upper bounds for the VaR when positive dependence information of the type $Y \preceq X$ is assumed.

**Theorem 3.1.** Assume that the random vector $Y$ has distribution $F_Y$ defined as in (3.1).

1. **(Lower bound)** If $Y \preceq \omega X$, then for any $\alpha \in (0, 1)$ we have

$$\text{VaR}_\alpha(X^*_d) \geq \sup_{L(\omega)} \sum_{j=1}^k n_j G^{-1}_j(u_j),$$

where $L(\omega) = \{ (u_1, \ldots, u_k) \in [0, \alpha]^k; \prod_{j=1}^k (1 - u_j) = 1 - \alpha \}$.

2. **(Upper bound)** If $Y \preceq \omega X$, then for any $\alpha \in (0, 1)$ we have

$$\text{VaR}_\alpha(X^*_d) \leq \inf_{U(\omega)} \sum_{j=1}^k n_j G^{-1}_j(u_j),$$

where $U(\omega) = \{ (u_1, \ldots, u_k) \in [\alpha, 1]^k; \prod_{j=1}^k u_j = \alpha \}$.

In order to prove the theorem we need the following lemma.

**Lemma 3.2.** We have that

$$\sup_{x_1 + \cdots + x_k = s} \prod_{j=1}^k \mathcal{G}_j(y_{ij}) = \sup_{x_1 + \cdots + x_k = s} \prod_{j=1}^k \mathcal{G}_j(y_{ij}),$$

where $\mathcal{G}_j = 1 - G_j, 1 \leq j \leq k$.

**Proof.** For any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ with $\sum_{j=1}^d x_j = s$ it is possible to define $z = (z_1, \ldots, z_d) \in \mathbb{R}^d$ as

$$z_i = \sum_{j=1}^k 1(i \in I_j) \cdot \left( \frac{1}{n_j} \sum_{r \in \mathcal{E}_j} x_r \right), \quad 1 \leq i \leq d.$$  

The vector $z$ has the components $z_i$ with index $i$ in the same subgroup $I_j$ identical and equal to the average of the $x_r$ with index belonging to the same $I_j$. It is straightforward to check that $\sum_{i=1}^d z_i = s$ and $\max_{i \in \omega} |x_i| \geq \max_{i \in \omega} |z_i|, 1 \leq j \leq k$. Since the product operator is coordinatewise increasing while the $\mathcal{G}_j$ are decreasing functions, we obtain that

$$\sup_{x_1 + \cdots + x_k = s} \prod_{j=1}^k \mathcal{G}_j(y_{ij}) \leq \sup_{y_1 + \cdots + y_k = s} \prod_{j=1}^k \mathcal{G}_j(y_{ij}).$$

The inverse inequality ($\geq$) is immediate. \hfill $\square$

**Proof of Theorem 3.1 (1).** By Proposition 2.2 we have

$$P(X^*_d \geq s) \geq \sup_{x \in \mathcal{E}(\omega)} F_Y(x) = \sup_{x \in \mathcal{E}(\omega)} \prod_{j=1}^k P(Y_j > x_j, i \in I_j)$$

$$= \sup_{x \in \mathcal{E}(\omega)} \prod_{j=1}^k \mathcal{G}_j(y_{ij}).$$

Applying Lemma 3.2 leads to

$$P(X^*_d \geq s) \geq \sup_{y_1 + \cdots + y_k = s} \prod_{j=1}^k \mathcal{H}_j(y_{ij}) = \sup_{y_1 + \cdots + y_k = s} \prod_{j=1}^k \mathcal{H}_j(y_{ij}),$$

where we set $\mathcal{H}_j(y_{ij}) = \mathcal{G}_j(y_{ij})/n_j, 1 \leq j \leq k$. From the duality principle given in Theorem 4.1 in Embrechts et al (2003), it follows that

$$\text{VaR}_\alpha(X^*_d) \geq \sup_{x_1 + \cdots + x_k = s} \sum_{j=1}^k H^{-1}_j(u_j) = \sup_{x_1 + \cdots + x_k = s} \sum_{j=1}^k n_j G^{-1}_j(u_j),$$

where $\mathcal{L}(\alpha) = \{ (u_1, \ldots, u_k) \in [0, 1]^k; \prod_{j=1}^k (1 - u_j) = 1 - \alpha \}$. The upper bound in (3.4) is now obtained by noting that the
constraints $\prod_{j=1}^{k}(1 - u_j) = 1 - \alpha$ and $u_j \in [0, 1]$ imply $u_j \leq \alpha, 1 \leq j \leq k$.

**Proof of Theorem 3.1 (2).** Using Proposition 2.2 we have

$$P(X_i^+ \leq s) \geq \sup_{x_1 + \cdots + x_k = s} \prod_{j=1}^{k} \min\{G_j(x_j)\}$$

$$= \sup_{k} \prod_{j=1}^{k} G_j\left(\min\{x_j\}\right).$$

(3.7)

Using the same argument as in the proof of Lemma 3.2, it follows that

$$P(X_i^+ \leq s) \geq \sup_{x_1 + \cdots + x_k = s} \prod_{j=1}^{k} G_j\left(\min\{x_j\}\right)$$

$$= \sup_{y_1 + \cdots + y_k = s} \prod_{j=1}^{k} G_j\left(\frac{y_j}{n_j}\right) = \sup_{y_1 + \cdots + y_k = s} \prod_{j=1}^{k} H_j(\alpha_j).$$

(3.8)

From Theorem 4.1 in Embrechts et al. (2003), it follows that

$$\text{VaR}_\alpha(X_i^+ \leq s) \leq \inf_{\mathcal{U}(\alpha)} \sum_{j=1}^{k} H_j^{-1}(u_j) = \inf_{\mathcal{U}(\alpha)} \sum_{j=1}^{k} n_j G_j^{-1}(u_j),$$

where $\mathcal{U}(\alpha) = \{(u_1, \ldots, u_k) \in [0, 1]^k; \prod_{j=1}^{k} u_j = \alpha\}$. The upper bound in (3.5) now follows by noting that the constraints $\prod_{j=1}^{k} u_j = \alpha$ and $u_j \in [0, 1]$ imply $u_j \geq \alpha, 1 \leq j \leq k$. \qed

3.1. Computation of the lower bound on the VaR

Under additional hypotheses on the marginal models $G_j$ it is possible to compute analytically the supremum in (3.4).

**Theorem 3.3.** Fix $\alpha \in (0, 1)$ and assume that the function $\psi_j : [\ln(1 - \alpha), 0] \to \mathbb{R}$ defined as

$$\psi_j(x) = G_j^{-1}(1 - e^x)$$

is continuous and convex for all $1 \leq j \leq k$. Then, we have that

$$\sup_{\mathcal{Q}(\alpha)} \sum_{j=1}^{k} n_j G_j^{-1}(u_j) = \max_{1 \leq j \leq k} \left\{ n_j G_j^{-1}(\alpha) + \sum_{i \neq j} n_i G_i^{-1}(0) \right\}.$$ 

Proof. Applying the transformation $u_j = 1 - e^{v_j}$, we obtain

$$\sup_{\mathcal{Q}(\alpha)} \sum_{j=1}^{k} n_j G_j^{-1}(u_j) = \sup_{\mathcal{V}(\alpha)} \sum_{j=1}^{k} n_j G_j^{-1}(1 - e^{v_j}),$$

(3.9)

where $\mathcal{V}(\alpha) = \{(v_1, \ldots, v_k) \in [\ln(1 - \alpha), 0]^k; \sum_{j=1}^{k} v_j = \ln(1 - \alpha)\}$ is a convex and compact set. By assumption, the function $\sum_{j=1}^{k} n_j G_j^{-1}(1 - e^{v_j})$ is continuous and convex and by the maximum principle it attains the maximum at an extreme point of $\mathcal{V}(\alpha)$, that is at a point $v$ such that $v_j = \ln(1 - \alpha)$ for some $j$ and $v_i = 0$ for every $i \neq j$. Hence, we have that

$$\sup_{\mathcal{V}(\alpha)} \sum_{j=1}^{k} n_j G_j^{-1}(1 - e^{v_j}) = \max_{1 \leq j \leq k} \left\{ n_j G_j^{-1}(1 - e^{\ln(1 - \alpha)}) + \sum_{i \neq j} n_i G_i^{-1}(1 - e^0) \right\},$$

which gives the desired result. \qed

The assumptions of Theorem 3.3 are easy to check and satisfied for some models of interest in quantitative risk management. We give some relevant examples below.

**Example 3.1 (Pareto marginal distributions).** First, we consider the case where all the $G_j$ are Pareto distributed with parameter $\theta_j > 0$, that is

$$G_j(x) = 1 - (1 + x)^{-\theta_j}, x > 0, 1 \leq j \leq k.$$ 

In this case the functions $G_j^{-1}(1 - e^x) = \exp(-x/\theta_j) - 1$ satisfy the assumptions of Theorem 3.3, and for a vector $X$ with $Y \leq \text{uo}$, we obtain

$$\text{VaR}_\alpha(X_i^+ \leq s) \leq \max_{1 \leq j \leq k} \left\{ n_j \left(1 - \alpha\right)^{-\theta_j} - 1 \right\}.$$ 

(3.10)

The bound (3.10) holds under the assumption $Y \leq \text{uo}$, which does not necessarily imply that $Y$ and $X$ have the same marginal distributions. Under the assumption that $X$ and $Y$ have also the same marginals, i.e.

$$F_i = G_j, \text{ if } i \in I_j,$$

it is natural to compare the lower bound in (3.10) with the unconstrained bound $\text{VaR}_\alpha(X_i^+)$ defined as

$$\text{VaR}_\alpha(X_i^+) = \inf \left\{ \text{VaR}_\alpha(X_i^+); F_X \in \mathcal{F}(F_1, \ldots, F_d) \right\}.$$ 

(3.11)

The bound $\text{VaR}_\alpha(X_i^+)$ is based only on the assumption of the knowledge of the marginal distributions of the risk portfolio $X$, without any information on its dependence structure. In Table 1 we report figures for the bounds (3.10) and (3.11) for a sum of $d$ homogeneous Pareto(2) random variables.

The bound $\text{VaR}_\alpha(X_i^+)$ can be computed in the Pareto case using Corollary 4.7 in Jakobsen et al. (2014). The lower bound obtained in (3.10) under a positive dependence assumption considerably improves the unconstrained bound for all $k < d$. Interestingly enough, the case $k = d = 8$, where the vector $X$ is assumed to be PUOD, exactly returns the unconstrained bound $\text{VaR}_\alpha(X_i^+)$, i.e. the PUOD condition alone does not increase the lower risk bound induced by marginals information only. The improvement implied by the extra dependence assumption decreases with increasing $k$, which corresponds to weaker positive dependence assumption (recall that $k = 1$ gives cocomonotonic $X_j$ while $k = 8$ corresponds to independent $Y_j$).
Example 3.2 (Exponential marginal distributions). We now assume that each $G_j$ is an Exponential distribution with positive parameter $\theta_j$, that is

$$G_j(x) = 1 - e^{-\theta_j x}, \ x > 0, 1 \leq j \leq k.$$  

Since $G_j^{-1}(1 - e^{-\lambda})$ is linear, from Theorem 3.3 we obtain

$$\text{VaR}_\alpha(X_j^*) \geq \max_{0 \leq j \leq k} \left\{ \frac{n_j}{n} \ln(1 - \alpha) \right\}.$$  

(3.12)

Again the lower bound obtained in (3.12) under a positive dependence assumption considerably improves the unconstrained bound for all $k < d$, see Table 2 (with a slight abuse of notation the case $k = 1$ indicates the inhomogeneous case when all the $X_j$ are comonotonic). Note that Theorem 3.3 allows to compute bounds for portfolios with different families of marginal distributions; see Table 5 below.

In case the conditions of Theorem 3.3 are not satisfied, the supremum in (3.4) might be attained in an internal point of $L(\alpha)$. We can rewrite (3.4) as

$$\text{VaR}_\alpha(X'_j) \geq \sup_{u_{i_1, \ldots, u_{i_k} \in [0, \alpha]} \ k} \ n_j G_j^{-1}(u_j) + n_k G_k^{-1} \left( 1 - \frac{1 - \alpha}{\Pi_{j=1}^k (1 - u_j)} \right).$$  

(3.13)

First order conditions for $\sup = \max$ in (3.13) are given by

$$\left\{ \begin{array}{l}
\frac{n_i (1 - u'_i)}{G^{-1}(u'_i)} = \frac{n_i (1 - \alpha)}{\Pi_{j=1}^k (1 - u'_j)}, \quad \text{for } i \in C; \\
\frac{n_i}{G^{-1}(u'_i)} = 0, \quad \text{for } i \notin C,
\end{array} \right.$$  

(3.14)

where we define $C = \{ j \in \{1, \ldots, k - 1\} : u'_j > 0 \}$ and $g_j = G_j'$. The system of equations in (3.14) generally needs to be solved numerically. To get solutions in closed form, we consider the simplified framework where the distributions $G_j$ are assumed to be all identical and each subgroup to have the same cardinality, i.e. $G_j = G$ and $n_j = \frac{d}{k}, 1 \leq j \leq k$. The set of equations in (3.14) then becomes

$$\left\{ \begin{array}{l}
\frac{1 - u'_i}{G^{-1}(u'_i)} = \frac{1 - \alpha}{\Pi_{j=1}^k (1 - u'_j)}, \quad \text{for } i \in C; \\
\frac{n_i}{G^{-1}(u'_i)} = 0, \quad \text{for } i \notin C.
\end{array} \right.$$  

(3.15)

The equations in (3.15) are solved by $u'_i = 1 - (1 - \alpha)^{\frac{1}{k}}, \ i \in C$. In the case the density of $G$ is monotone in $[G^{-1}(0), G^{-1}(1)]$, this solution is unique. Considering all the possible cardinalities of $C$ in (3.15), we collect a set of $k$ candidate vectors, one of which gives (in case uniqueness holds) $\sup = \max$ in (3.13). For this homogeneous framework, the lower bound for $\text{VaR}$ in (3.4) becomes

$$\text{VaR}_\alpha(X'_d) \geq \max_{0 < j < k} \left\{ \frac{d(k - j)}{k} G^{-1} \left( 1 - \frac{1}{(1 - \alpha)^{\frac{1}{k}}} \right) + \frac{d(j)}{k} G^{-1}(0) \right\}. $$  

(3.16)

We remark that the inequality in (3.16) always holds in an homogeneous framework and does not need any extra assumption on the distribution $G$. In case the density of $G$ is not monotone on $[G^{-1}(0), G^{-1}(1)]$, the bound given in (3.16) might however be improved by finding other different solutions to (3.15) by involving more advanced techniques that fall beyond the scope of the present paper; see Embrechts and Puccetti (2006) on the applicability of multivariate optimization to a similar problem.

Example 3.3 (Homogeneous Gamma portfolio). We compute the bound (3.16) for an homogeneous portfolio of $d = 16$ Gamma distributions; figures are reported in Table 3. As one can see from this table, the lower bound tends to deteriorate with increasing dimensions. For relatively small numbers of independent subgroups, intermediate positive dependence restrictions typically improve the lower bounds on $\text{VaR}$, while a positive upper orthant dependence assumption ($k = d$) does not provide any improvement. In Table 3 the unconstrained bound $\text{VaR}_\alpha(X'_d)$ is obtained using the rearrangement procedure described in Embrechts et al. (2013).  

3.2. Computation of the upper bound on the VaR

We can rewrite (3.5) as

$$\text{VaR}_\alpha(X'_d) \leq \inf_{u_{i_1, \ldots, u_{i_k} \in [0, \alpha]}} \ k \ n_j G_j^{-1}(u_j) + n_k G_k^{-1} \left( \frac{\alpha}{\Pi_{j=1}^k (1 - u_j)} \right).$$  

(3.17)

First order conditions for $\inf = \min$ in (3.17) are given by

$$\left\{ \begin{array}{l}
\frac{n_i u'_i}{G^{-1}(u'_i)} = \frac{n_i}{\Pi_{j=1}^k (1 - u'_j)}, \quad \text{for } i \in B; \\
\frac{n_i}{G^{-1}(u'_i)} = 0, \quad \text{for } i \notin B,
\end{array} \right.$$  

(3.18)

where we denote $B = \{ j \in \{1, \ldots, k - 1\} : u'_j < 1 \}$. In general, the set of equations in (3.18) needs to be solved numerically. To get solutions in closed form, we consider a simplified framework, where we assume that all the $G_j$ are identical, and each subgroup has the same cardinality. In practice, we set $G = F$ and $n_j = d/k$, $1 \leq j \leq k$. The equations in (3.18) then become

$$\left\{ \begin{array}{l}
\frac{u'_i}{G^{-1}(u'_i)} = \frac{\alpha}{\Pi_{j=1}^k (1 - u'_j)}, \quad \text{for } i \in B; \\
\frac{n_i}{G^{-1}(u'_i)} = 0, \quad \text{for } i \notin B,
\end{array} \right.$$  

(3.19)

The equations in (3.19) are solved by $u'_i = \frac{\alpha}{k}, \ j \in B$. In the case the density of $G$ is monotone in $[G^{-1}(1), G^{-1}(1)]$, this solution is unique. Considering all the possible cardinalities of $B$ in (3.19), we collect a set of $k$ candidate vectors, one of which
Table 1: Values (rounded) for the VaR bound in (3.10) (denoted by VaR\(^{\scriptscriptstyle lb}\)) for an homogeneous portfolio with \(d\) Pareto(2) risks, \(k\) subgroups, \(d/k\) variables in each subgroup and dependence assumption \(Y \leqso X\). The unconstrained bound \(\text{VaR}_{uo}(X_d)\) in (3.11) is also reported. In this and the forthcoming tables, the indication of the random variable \(X_d^j\) is omitted.

<table>
<thead>
<tr>
<th>(d = 8)</th>
<th>(k = 1)</th>
<th>(k = 2)</th>
<th>(k = 4)</th>
<th>(k = 8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha = 0.990)</td>
<td>9.00</td>
<td>72.00</td>
<td>9.00</td>
<td>6.00</td>
</tr>
<tr>
<td>(\alpha = 0.995)</td>
<td>13.14</td>
<td>105.14</td>
<td>13.14</td>
<td>52.57</td>
</tr>
<tr>
<td>(\alpha = 0.999)</td>
<td>30.62</td>
<td>244.98</td>
<td>30.62</td>
<td>122.49</td>
</tr>
</tbody>
</table>

Table 2: Values (rounded) for the VaR bound in (3.12) (denoted by VaR\(^{\scriptscriptstyle lb}\)) for an inhomogeneous portfolio with \(d/2\) Exp(2) risks and \(d/2\) Exp(4) risks, \(k\) subgroups, \(d/k\) variables in each subgroup and dependence assumption \(Y \leqso X\). The unconstrained bound \(\text{VaR}_{uo}(X_d)\) is also reported.

<table>
<thead>
<tr>
<th>(d = 8)</th>
<th>(k = 1)</th>
<th>(k = 2)</th>
<th>(k = 4)</th>
<th>(k = 8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha = 0.990)</td>
<td>2.30</td>
<td>13.82</td>
<td>2.30</td>
<td>9.21</td>
</tr>
<tr>
<td>(\alpha = 0.995)</td>
<td>2.65</td>
<td>15.89</td>
<td>2.65</td>
<td>10.60</td>
</tr>
<tr>
<td>(\alpha = 0.999)</td>
<td>3.45</td>
<td>20.72</td>
<td>3.45</td>
<td>13.82</td>
</tr>
</tbody>
</table>

Table 3: Values (rounded) for the VaR bound in (3.16) (denoted by VaR\(^{\scriptscriptstyle lb}\)) for an homogeneous portfolio of \(d\) Gamma(3,1/2) risks and dependence assumption \(Y \leqso X\). The unconstrained bound \(\text{VaR}_{uo}(X_d)\) is also reported. The parameterization of the Gamma distribution used can be found in Example 4.4.

<table>
<thead>
<tr>
<th>(d = 16)</th>
<th>(k = 1)</th>
<th>(k = 2)</th>
<th>(k = 4)</th>
<th>(k = 8)</th>
<th>(k = 16)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha = 0.990)</td>
<td>23.47</td>
<td>67.25</td>
<td>23.47</td>
<td>42.58</td>
<td>23.47</td>
</tr>
<tr>
<td>(\alpha = 0.995)</td>
<td>23.70</td>
<td>74.19</td>
<td>23.70</td>
<td>46.53</td>
<td>23.70</td>
</tr>
<tr>
<td>(\alpha = 0.999)</td>
<td>23.93</td>
<td>89.83</td>
<td>23.93</td>
<td>55.31</td>
<td>23.93</td>
</tr>
</tbody>
</table>

Example 3.4 (Homogeneous Pareto portfolio). We now consider the case that each \(G_i\) is a Pareto(2) distribution. Since the Pareto distribution is unbounded from above (i.e. \(G^{-1}(1) = \infty\)), the min in (3.20) is attained when \(j = k\), therefore (3.20) simplifies to

\[
\text{VaR}_d(X_d^j) \leq d G^{-1}(\alpha_j^\frac{1}{j}).
\]  

The bound (3.21) holds under the assumption \(Y \leqso X\), which does not necessarily imply that \(Y\) and \(X\) have the same marginal distributions. Under the extra assumption (3.3) that \(X\) and \(Y\) have also the same marginals, it is natural to compare the upper bound in (3.20) with the unconstrained bound \(\text{VaR}_{uo}(X_d)\) defined as

\[
\overline{\text{VaR}}_{uo}(X_d) = \sup \{\text{VaR}_{uo}(X_d): F_X \in \mathcal{F}(F_1, \ldots, F_d)\},
\]  

where \(F_i = G_j\) if \(i \in I_j\). The bound \(\overline{\text{VaR}}_{uo}(X_d)\) is based only on the assumption of the knowledge of the marginal distributions of the risk portfolio \(X\), and is easily obtained in this homogeneous case from Proposition 4 in Embrechts et al. (2013). In Table 3 we report figures for the bounds (3.21) and (3.22) for a sum of \(d\) homogeneous Pareto(2) random variables. Our results indicate that the bound given in (3.20) with positive dependence information improves the unconstrained upper bound only for \(k = 1\) (since in this the case the \(X_d^j\) are assumed to be comonotonic) and \(k = 2\) (when the bound is known to be sharp). This is in concordance with our general findings that positive dependence helps to improve the lower unconstrained bound on VaR, but does not help to improve the corresponding upper bound.
3.3. General lower and upper bounds

If \( Y \) has a set of marginal distributions for which the assumptions of Theorem 3.5 are not satisfied, it is however important to notice that from (3.4) the inequality

\[
\text{VaR}_{\alpha}(X_d) \geq \max_{1 \leq j \leq k} \left\{ n_j G_j^{-1}(\alpha) + \sum_{i \neq j} n_i G_i^{-1}(0) \right\}
\]

always holds in case \( Y \leq_{wo} X \). Analogously, since the vector \( u = (\alpha^{1/k}, \ldots, \alpha^{1/k}) \) is admissible in (3.5), we have

\[
\text{VaR}_\alpha(X_d) \leq \sum_{j=1}^k n_j G_j^{-1}(\alpha^{1/k}),
\]

when \( Y \leq_{lo} X \). The inequalities in (3.22) and in (3.24) might yield improved bounds for general portfolios and are straightforward to compute. Both bounds hold if \( Y \leq_{cw} X \) is assumed, implying also that \( X \) and \( Y \) have the same marginals; see Table 6. In Tables 5–8 we report the bounds (3.23) and (3.24) for some inhomogeneous portfolios of Pareto, Exponential and LogNormal marginal distributions. In these tables, the lower bound in (3.23) is shown to improve the corresponding unconstrained one for \( k = 2, 4 \), while the upper bound in (3.24) is useful to reduce model uncertainty in the case \( k = 2 \). In Tables 5–8 the unconstrained bounds \( \text{VaR}_\alpha \) and \( \text{VaR}_\alpha \) are computed according to Corollary 4.7 in Jakobson et al. (2014) (when possible) or via the RA technique described in Embrechts et al. (2013).

4. Reducing dependency spreads of convex risk measures

Recall that \( X = (X_1, \ldots, X_d) \) represents a risk portfolio having marginal distributions \( F_1, \ldots, F_d \). To find reduced dependence uncertainty spreads on convex risk measures, throughout this section we assume

\[
Y \leq_{wcs} X,
\]

where the vector \( Y \) is a random vector having the same univariate marginal distributions as \( X \). We also denote by \( X^c = (X_1^c, \ldots, X_d^c) \) a comonotonic vector having the same univariate marginal distributions as \( X \), that is having joint distribution

\[
F_{X^c}(x_1, \ldots, x_d) = \min\{F_1(x_1), \ldots, F_d(x_d)\}.
\]

We shall make use of the convex order \( X \lesssim_{cx} Y \) between random variables \( X, Y \) defined by \( X \leq_{cx} Y \) iff \( \mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)] \) for all convex functions \( f: \mathbb{R} \rightarrow \mathbb{R} \) such that the expectations exist.

Let \( Y^c_j = \sum_{d=1}^d Y_j, X^c_j = \sum_{d=1}^d X_j \) and \( X^c = \sum_{j=1}^d X^c_j \). The dependence ordering result given in Theorem 2.1 in Rüschendorf (2004) and a classical result in Meilijson and Nádas (1979) imply that

\[
Y^c_j \lesssim_{cx} X^c_j \lesssim_{cx} X^c.
\]

Further, law-invariant, convex risk measures satisfying the so-called Fatou property are consistent with respect to the convex order; see Corollary 4.65 in Föllmer and Schied (2004) and Theorem 4.3 in Bäuerle and Müller (2006). Therefore (4.2) implies the following result.

**Theorem 4.1.** Let \( \rho \) be a law-invariant, convex risk measure satisfying the Fatou property. Assume that the random vectors \( X \) and \( Y \) have the same marginal distributions and \( Y \leq_{wcs} X \). Then

\[
\rho(Y^c_j) \leq \rho(X^c_j) \leq \rho(X^c).
\]

The Fatou property is a standard weak continuity condition which is satisfied by all the risk measures used in the following.

**Remark 4.1.** Theorem 4.1 holds under the slightly weaker assumptions \( Y \leq_{sm} X \) or \( Y \leq_{dxc} X \), where \( \leq_{dxc} \) is the directionally convex order; see Section 3.4.14 in Denuit et al. (2005) or Remark 6.27 (b) in Rüschendorf (2013). Since \( \leq_{wcs} \) has a more direct interpretation as a positive dependence ordering and can be checked in several functional models (see Rüschendorf 2004), we state Theorem 4.1 under this ordering condition. For the non convex risk measure VaR the conclusion in (4.3) is not true; see the several examples given in Embrechts et al. (2014a).

The upper bound \( \rho(X^c_j) \) in (4.3) corresponds to the case of maximally correlated risks and can be easily computed in the examples to follow. Furthermore, once fixed the marginal distributions of \( X \), the upper bound \( \rho(X^c_j) \) in (4.3) holds without any dependence assumption, therefore we have

\[
\rho(X^c_j) = \overline{\rho}(X^c_j).
\]

Again, the dependence assumption \( Y \leq_{wcs} X \) has only influence on the lower bound \( \rho(Y^c_j) \) which is generally strictly bigger than the unconstrained bound \( \rho(X^c_j) \) obtained without any positive dependence information.
To keep the connection with Section 3, in the examples below we assume that $Y$ has the joint distribution defined in (3.1). Since in this section $X$ is assumed to have the same marginals as $Y$, this implies that

$$F_i = G_j, \text{ if } i \in I_j.$$ 

In the remainder, we study portfolios of Gamma distributed risks since the Gamma distribution allows to obtain all the quantities in (4.3) in closed form. For general portfolios: the unconstrained lower bound $\rho(X'_d)$ can be computed using the numerical procedures described in [Puccetti and Rüschendorf (2015)] (for entropic risk measures) and in [Puccetti (2013)] (for the Expected Shortfall); the constrained lower bound $\rho(Y'_d)$ can be generally computed numerically or by Monte Carlo simulation; the upper bound $\rho(X''_d) = \overline{\rho}(X'_d)$ is easily available numerically considering that $X''$ is comonotonic.

**Example 4.1 (Expected Shortfall).** We compute the bounds in (4.3) when $\rho = ES_{\alpha}$ for a portfolio of Gamma distributed random variables. For a random variable $X$ with $\mathbb{E}[X] < \infty$, the Expected Shortfall (ES) at confidence level $\alpha \in [0, 1)$ is defined as

$$ES_{\alpha}(X) = \frac{1}{1 - \alpha} \int_0^\infty \text{VaR}_{\alpha}(X) \, dq.$$ 

If a random variable $X$ follows a $F_{\alpha,s} = \text{Gamma}(\alpha, s)$ distribution with positive parameters $\alpha, s$ and density function

$$f_{\alpha,s}(x) = \frac{1}{s^\alpha \Gamma(a)} x^{a-1} e^{-x/s},$$ 

it is elementary to show that

$$ES_{\alpha}(X) = \frac{1}{1 - \alpha} \frac{\Gamma(a + 1)s}{\Gamma(a)} \left( F_{\alpha+1,a-s}^{-1}\left( F_{\alpha,s}(\alpha) \right) \right).$$ (4.4) 

Assuming that $G_j = \text{Gamma}(\alpha_j, s)$, $1 \leq j \leq k$ in (3.1), it follows that $\sum_{i \in I_j} Y_i$ is again a Gamma random variable with distribution $F_{\alpha_i,s_i}^{-1} 1 \leq j \leq k$. Setting $n_j = d/k, 1 \leq j \leq k$ and taking the convolution of Gamma with identical scale parameter, we obtain that $Y'_d$ has Gamma distribution $F_{\alpha_1,s_1}^{-1} F_{\alpha_2,s_2}^{-1} \cdots F_{\alpha_k,s_k}^{-1}$. From (4.3) finally find that the constrained lower bound $ES_{\alpha}(Y'_d)$ in (4.3) is given by

$$ES_{\alpha}(Y'_d) = \frac{d}{k \left(1 - \alpha \right)} \frac{\Gamma(1 + \sum_{j=1}^k a_j)}{\Gamma(\sum_{j=1}^k a_j)} \left( F_{\alpha,\sum_{j=1}^k a_j,ds/k}^{-1}(F_{\alpha_1,s_1}^{-1} F_{\alpha_2,s_2}^{-1} \cdots F_{\alpha_k,s_k}^{-1}(\alpha)) \right).$$ 

The unconstrained lower bound $ES_{\alpha}(X'_d)$ defined in (1.2) with $\rho = ES_{\alpha}$ is very well approximated by the sum of the first mo-
mment $\mu = 12$ of the Gamma marginals under study, that is

$$E_{\rho}(X_d^+)|_{\rho} = E_{\rho}(\mu) = \mu.$$ 

Using the mixability detection procedure in [Puccetti and Wang (2014)], it is in fact possible to show that the set of Gamma distributions used in Table 8 conditional to the intervals $[0, F^{-1}_j(1-10^{-7})]$, are very close to being jointly mixable, that is there exists a random vector $(W_1,\ldots,W_d)$ having the prescribed conditional marginal distributions under which

$$P(W_1 + \cdots + W_d \in [\mu - 10^{-3},\mu + 10^{-3}]) = 1.$$ 

Note that this also directly implies that $\rho(X_d^+) \approx \rho(\mu)$ in the forthcoming examples where other convex risk measures $\rho$ are considered. We refer to [Puccetti and Wang (2014)] and the references therein for more details on joint mixability.

Since $E_{\rho}$ is a comonotonic additive convex risk measure, the upper bound $E_{\rho}(X_d^+)$ is equal to the sum of marginal expected shortfalls, i.e. we have

$$E_{\rho}(X_d^+) = \sum_{j=1}^d E_{\rho}(X_j).$$

In Table 8 we report the bounds $E_{\rho}(X_d^+)$, $E_{\rho}(Y_d^+)$, $E_{\rho}(X_d^-) = E_{\rho}(Y_d^-)$ for a portfolio of inhomogeneous Gamma distributions as well as the implied DU-spreads. With a slight abuse of notation the case $k = 1$ indicates in the forthcoming tables the case when all the $X_j$ are comonotonic.

Example 4.2 (Entropic risk measure). For a random variable $X$ with $E[|X|] < \infty$, the Entropic Risk Measure (ERM) with (risk aversion) parameter $\beta > 0$ is defined as

$$\text{ERM}_\beta(X) = \frac{1}{\beta} \log E[e^{\beta X}],$$

provided the expectation is finite. The entropic risk measure is well known to be a convex risk measure (see for instance Föllmer and Schied [2004]). For a random variable $X$ with distribution Gamma$(a, s)$, it follows immediately by the definition of the moment generating function $(M_X(t) = E[e^{tX}]$ that:

$$\text{ERM}_\beta(X) = \frac{1}{\beta} \log (1 - s\beta)^{-a} \quad \text{for} \quad 0 < \beta < \frac{1}{s}.$$ 

We already know from Example 4.1 that $Y_d^+$ has Gamma distribution $F_{\sum_{j=1}^d a_j, ds/k}$ hence we obtain

$$\text{ERM}_\beta(Y_d^+) = \frac{1}{\beta} \log \left(1 - \frac{ds}{k}\right)^{-a} \quad \text{for} \quad 0 < \beta < \frac{k}{ds}.$$ 

From Theorem 4.1 the upper bound in 4.3 for the entropic risk measure is obtained when all the $X_j$ are comonotonic, that is

$$\text{ERM}_b(X_d^+) = \text{ERM}_b(X_d^+) = \frac{1}{\beta} \log E \left[ e^{\rho(n_1 F_a^{-1}_i(U) + \cdots + n_k F_a^{-1}_i(U))} \right],$$

where $F_a^{-1}_i$ is the Gamma inverse cumulative distribution function with parameters $a$, $s$ and $U$ is a random variable uniformly distributed on $(0, 1)$. Similarly to what discussed in Example 4.1 we obtain $\text{ERM}_b(X_d^+) \approx 12$. In Table 9 we report the bounds $\text{ERM}_b(X_d^+)$, $\text{ERM}_b(Y_d^+)$ and $\text{ERM}_b(Y_d^-)$ for a portfolio of inhomogeneous Gamma distributions as well as the corresponding dependence uncertainty spreads.

Example 4.3 (Expectiles). We compute the bounds in 4.3 using expectiles as a risk measure, $\rho = \rho_e$, for a portfolio of Gamma distributed random variables. Expectiles are attracting increasing interest in the literature on risk measures since they are the only coherent risk measure with the additional property of being elicitable; for more details on this see for instance Ziegel (2014) and Bellini and Bignozzi (2014). For a random variable $X$ with $E[|X|] < \infty$, the expectile $e_{\rho_e}$, computed at confidence level $p \in (0, 1)$, is defined as the unique solution to

$$p E[X - e_{\rho_e}(X)] = (1 - p) E[X - e_{\rho_e}(X)],$$

where $x_+ = \max(0,x)$ and $x_- = \max(0,-x)$. When $p \geq 1/2$, expectiles are coherent risk measures. Expectiles are generally not available in closed form and need to be computed numerically.

Again from Theorem 4.1 the upper bound in 4.3 for the expectiles is obtained when all the $X_j$ are comonotonic. Hence we have to find numerically the unique solution to

$$p E[(n_1 F_a^{-1}_i(U) + \cdots + n_k F_a^{-1}_i(U)) - e_{\rho_e}(n_1 F_a^{-1}_i(U) + \cdots + n_k F_a^{-1}_i(U)))] = (1 - p) E[(n_1 F_a^{-1}_i(U) + \cdots + n_k F_a^{-1}_i(U)) - e_{\rho_e}(n_1 F_a^{-1}_i(U) + \cdots + n_k F_a^{-1}_i(U))].$$

The computation of $e_{\rho_e}(X_d^+)$ is analogous while, similarly to what discussed in Example 4.1 we obtain $e_{\rho_e}(X_d^+) \approx 12$. In Table 10 we report the bounds 4.3 for a portfolio of inhomogeneous Gamma distributions as well as the unconstrained bounds $e_p(X_d^+)$ and $e_p(Y_d^+)$.

5. Using negative dependence assumptions

We have introduced a class of positive dependence restrictions allowing to determine analytically improved lower risk bounds for the risk of a joint portfolio compared to the case where only marginals information is available. In a variety of examples of interest in quantitative risk management, positive dependence assumptions added on top of marginals information allow to increase the best-possible lower risk $\rho(X_d^+)$ and therefore reduce the implied dependence uncertainty spread $\bar{\rho}(X_d^+) - \rho(X_d^+)$. However, positive dependence assumptions seem to be ineffective in reducing the worst-possible estimate $\bar{\rho}(X_d^+)$ apart from
some trivial cases of limited applicability. Some kind of negative dependence allows to construct worst-case VaR distributions with VaR value bigger than in the comonotonic case (see [Wang and Wang 2011], [Embrechts et al. 2013]) and approaching asymptotically the worst ES bound (see [Puccetti et al. 2013], [Puccetti and Rüschendorf 2014]).

It seems intuitively clear that one has to assume some negative dependence constraints in order to reduce the upper bound on a risk measure. A negative dependence condition on $X$ is obtained assuming that

$$X \leq_{wcs} Y.$$  \hfill (5.1)

If $Y$ has the distribution specified in \((5.1)\), then \((5.1)\) does not pose essential restriction on the dependence structure within the groups $I_j$ but affects the joint dependence of the groups which is assumed to be more negatively dependent compared to independent groups. Similarly to Theorem 4.1, the assumption $X \leq_{wcs} Y$ implies the following result.

**Theorem 5.1.** Let $\rho$ be a law-invariant, convex risk measure satisfying the Fatou property. Assume that the random vectors $X$ and $Y$ have the same marginal distributions and $X \leq_{wcs} Y$. Then

$$\rho(X^d) \leq \rho(Y^d) \leq \rho(X^d).$$  \hfill (5.2)

The upper bound $\rho(Y^d)$ in \((5.2)\) is in general a strong improvement of the comonotonic upper bound $\rho(X^d)$. The assumption $X \leq_{wcs} Y$ may be realistic in hierarchical insurance models where some branches (groups) of insurance companies are approximately independent or possibly negatively dependent. For example bad weather type insurance (like hail, heavy rain, storm, etc.) could be supposed realistically to be negatively dependent to hot weather type insurances (like draught, fire, thunderstorm, etc.). Some more detailed study of this type of applications is planned for a future study.

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**References**


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### Table 8: Values (rounded) for the bounds $ES_n(Y^d)$ (denoted by $ES_n^b$), $ES_n(X^d)$ and $ES_n(X^d)$ for an inhomogeneous portfolio with $d/2$ Gamma(2,1/2) risks and $d/2$ Gamma(4,1/2) risks, $k$ subgroups, $n = d/k$ variables in each subgroup and dependence assumption $Y \leq_{wcs} X$. In this and in the forthcoming tables, the reduction of DU-spreads implied by additional dependence assumption is also reported.

<table>
<thead>
<tr>
<th>$d = 8$</th>
<th>unconstrained</th>
<th>(k = 1)</th>
<th>(k = 2)</th>
<th>(k = 4)</th>
<th>(k = 8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 0.990$</td>
<td>12.00</td>
<td>38.27</td>
<td>26.27</td>
<td>38.27</td>
<td>-100%</td>
</tr>
<tr>
<td>$\beta = 0.10$</td>
<td>12.00</td>
<td>15.22</td>
<td>3.22</td>
<td>15.22</td>
<td>-100%</td>
</tr>
<tr>
<td>$\beta = 0.15$</td>
<td>12.00</td>
<td>18.14</td>
<td>6.14</td>
<td>18.14</td>
<td>-100%</td>
</tr>
<tr>
<td>$\beta = 0.20$</td>
<td>12.00</td>
<td>23.80</td>
<td>11.80</td>
<td>23.80</td>
<td>-100%</td>
</tr>
</tbody>
</table>

---

### Table 9: Values (rounded) for the bound $ERM_{\rho}(Y^d)$ (denoted by $ER_{\rho}^b$), $ERM_{\rho}(X^d)$ and $ERM_{\rho}(X^d)$ for an inhomogeneous portfolio with $d/2$ Gamma(2,1/2) risks and $d/2$ Gamma(4,1/2) risks, $k$ subgroups, $n = d/k$ variables in each subgroup and dependence assumption $Y \leq_{wcs} X$.

<table>
<thead>
<tr>
<th>$d = 8$</th>
<th>unconstrained</th>
<th>(k = 1)</th>
<th>(k = 2)</th>
<th>(k = 4)</th>
<th>(k = 8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 0.990$</td>
<td>12.00</td>
<td>38.27</td>
<td>26.27</td>
<td>38.27</td>
<td>-100%</td>
</tr>
<tr>
<td>$\beta = 0.10$</td>
<td>12.00</td>
<td>15.22</td>
<td>3.22</td>
<td>15.22</td>
<td>-100%</td>
</tr>
<tr>
<td>$\beta = 0.15$</td>
<td>12.00</td>
<td>18.14</td>
<td>6.14</td>
<td>18.14</td>
<td>-100%</td>
</tr>
<tr>
<td>$\beta = 0.20$</td>
<td>12.00</td>
<td>23.80</td>
<td>11.80</td>
<td>23.80</td>
<td>-100%</td>
</tr>
</tbody>
</table>
Table 10: Values (rounded) for the bound $\varepsilon_p(Y^\Delta)$ (denoted by $\varepsilon_{plb}^p$), $\varepsilon_p(X^\Delta)$ and $\varepsilon_p(X^\Delta)$ for an inhomogeneous portfolio with $d/2$ Gamma(2,1/2) risks and $d/2$ Gamma(4,1/2) risks, $k$ subgroups, $n = d/k$ variables in each subgroup and dependence assumption $Y \leq_{wes} X$.

<table>
<thead>
<tr>
<th>$d = 8$</th>
<th>unconstrained</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
<th>$k = 4$</th>
<th>$k = 8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon_p$</td>
<td>$\varepsilon_p$</td>
<td>DU-S</td>
<td>$\varepsilon_{plb}^p$</td>
<td>DU-S</td>
<td>$\varepsilon_{plb}^p$</td>
</tr>
<tr>
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<td>12.00</td>
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<td>6.71</td>
<td>18.71</td>
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</tr>
<tr>
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<td>21.34</td>
<td>9.34</td>
<td>21.34</td>
<td>$-100%$</td>
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<tr>
<td>$p = 0.99$</td>
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<td>27.52</td>
<td>15.52</td>
<td>27.52</td>
<td>$-100%$</td>
</tr>
</tbody>
</table>


