General volatility smile asymptotics with bounded maturity

Jacopo Corbeta

Università degli Studi di Milano-Bicocca

Joint work with Francesco Caravenna (Milano-Bicocca)

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Outline

1. Preliminaries and notations
2. From option price to implied volatility
3. From tail probability to option price
4. Examples
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1. Preliminaries and notations

2. From option price to implied volatility

3. From tail probability to option price

4. Examples
The setting

- Log-price process \((X_t)_{t \geq 0}\) with \(X_0 = 0\) (risk-neutral measure)
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- European call and put (maturity \(t > 0\), log-strike \(\kappa \in \mathbb{R}\))

\[
c(\kappa, t) = \mathbb{E}[(e^{X_t} - e^{\kappa})^+] , \quad p(\kappa, t) = \mathbb{E}[(e^{\kappa} - e^{X_t})^+] .
\]
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- Right- and left-tail probability:
  
  \[
  \overline{F}_t(\kappa) := \mathbb{P}(X_t > \kappa) , \quad F_t(\kappa) := \mathbb{P}(X_t \leq \kappa) .
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\bar{F}_t(\kappa) := P(X_t > \kappa) , \quad F_t(\kappa) := P(X_t \leq \kappa) .
\]

- We take limits along an arbitrary family \((\kappa_s, t_s)_{s \geq 0}\).
- We assume that \(\kappa_s \geq 0\) and give results for both \(\kappa_s\) and \(-\kappa_s\).
Recall the standard Black&Scholes formula for a European call:

$$C_{BS}^{\sigma}(\kappa, t) = \Phi(d_1) - e^{\kappa} \Phi(d_2),$$

$$d_1 = -\frac{\kappa}{\sigma \sqrt{t}} + \frac{\sigma \sqrt{t}}{2}, \quad d_2 = -\frac{\kappa}{\sigma \sqrt{t}} - \frac{\sigma \sqrt{t}}{2}.$$
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\]

**Definition**

The implied volatility \( \sigma_{\text{imp}}(\kappa, t) \) of the model is the unique value of \( \sigma \in [0, \infty) \) such that

\[
C_{BS}^\sigma(\kappa, t) = c(\kappa, t).
\]
Aim of the paper

Find an explicit link between tail probability and implied volatility

when $\kappa \to \infty$ with bounded $t$, or $t \to 0$ with arbitrary $\kappa$. 

Remark

We require $c(\kappa, t) \to 0$, $p(-\kappa, t) \to 0$, with $t$ bounded from above.
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This gathers

- $\kappa \to \infty$ and $t \to \bar{t} \in (0, \infty)$;
- $\kappa \to \infty$ and $t \to 0$;
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Remark

We require

\[ c(\kappa, t) \to 0, \quad p(-\kappa, t) \to 0, \]

with \( t \) bounded from above.
Comparison with literature

- Lee (2004): limsup in the case $|\kappa| \to \infty$ for fixed $t$
- Benaim and Friz (2009): case $|\kappa| \to \infty$ for fixed $t$
- Gao and Lee (2014): sharper estimates in the regime $\kappa \not\to 0$
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Heuristics

For fixed $\kappa > 0$ and $t \to 0$, or $\kappa \to \infty$ and fixed $t > 0$

$$c(\kappa, t) = C_{BS}^{\sigma_{imp}(\kappa, t)}(\kappa, t) \approx e^{-\frac{\kappa^2}{2\sigma_{imp}(\kappa, t)t}}$$

($\star$)
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Inverting this relation gives

$$\sigma_{imp}(\kappa, t) \sim \frac{\kappa}{\sqrt{2t} \sqrt{-\log c(\kappa, t)}} \tag{**}$$
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$$c(\kappa, t) = C_{BS}^{\sigma_{imp}(\kappa, t)}(\kappa, t) \approx e^{\frac{-\kappa^2}{2\sigma_{imp}(\kappa, t)t}} \quad (\star)$$

Inverting this relation gives

$$\sigma_{imp}(\kappa, t) \sim \frac{\kappa}{\sqrt{2t} \sqrt{-\log c(\kappa, t)}} \quad (\star\star)$$

Relations $(\star)$-$(\star\star)$ are not always true, when $\kappa \to \infty$.

In complete generality, we can state the following result. Define

$$D(x) = \frac{\varphi(x)}{x} - \Phi(-x)$$

with

$$\begin{align*}
\varphi(z) &:= e^{-\frac{z^2}{2}} \sqrt{2\pi} \\
\Phi(x) &:= \int_{-\infty}^{x} \varphi(z) \, dz
\end{align*}$$
Theorem 1: from option price to implied volatility

Consider an arbitrary family of \((\kappa, t) = ((\kappa_s, t_s))_{s \geq 0}\) such that \(c(\kappa, t) \to 0\) (i.e. in all the previously stated regimes).
Theorem 1: from option price to implied volatility

Consider an arbitrary family of \((\kappa, t) = ((\kappa_s, t_s))_{s \geq 0}\) such that 
\(c(\kappa, t) \to 0\) (i.e. in all the previously stated regimes).

- Case of \(\kappa\) bounded away from zero (i.e. \(\lim \inf \kappa > 0\)).

\[
\sigma_{\text{imp}}(\kappa, t) \sim \left(\sqrt{\frac{-\log c(\kappa, t)}{\kappa}} + 1 - \sqrt{\frac{-\log c(\kappa, t)}{\kappa}}\right)\sqrt{\frac{2\kappa}{t}}
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- **Case of \(\kappa \to 0\)**

  \[
  \sigma_{\text{imp}}(\kappa, t) \sim \frac{1}{D^{-1}\left(\frac{c(\kappa, t)}{\kappa}\right)} \frac{\kappa}{\sqrt{t}}
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- **Case of \(\kappa = 0\)**

  \[ \sigma_{\text{imp}}(0, t) \sim \sqrt{2\pi} \frac{c(0, t)}{\sqrt{t}} \]
Making the formulas more explicit

If \(-\log c(\kappa, t) / \kappa \to \infty\), for \(\kappa\) bounded away from zero we get (⋆⋆)

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\sigma_{\text{imp}}(\kappa, t) \sim \frac{\kappa}{\sqrt{2t \left( -\log c(\kappa, t) \right)}}.
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If \(-\log c(\kappa, t) \over \kappa \rightarrow \infty\), for \(\kappa\) bounded away from zero we get (⋆⋆)

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\sigma_{\text{imp}}(\kappa, t) \sim \frac{\kappa}{\sqrt{2t (-\log c(\kappa, t))}}.
\]

If \(\kappa \rightarrow 0\)

\[
\sigma_{\text{imp}}(\kappa, t) \sim \begin{cases} 
\frac{\kappa}{\sqrt{2t (-\log(c(\kappa, t)/\kappa))}} & \text{if } \frac{c(\kappa, t)}{\kappa} \rightarrow 0; \\
\frac{\kappa}{D^{-1}(a) \sqrt{t}} & \text{if } \frac{c(\kappa, t)}{\kappa} \rightarrow a \in (0, \infty); \\
\sqrt{2\pi} \frac{c(\kappa, t)}{\sqrt{t}} & \text{if } \frac{c(\kappa, t)}{\kappa} \rightarrow \infty.
\end{cases}
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Heuristic

Crucial Observation

Under mild assumption, for $\kappa > 0$, as $t \downarrow 0$,

$$c(\kappa, t) \approx P(X_t > \kappa) = \overline{F}_t(\kappa).$$
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Crucial Observation

Under mild assumption, for $\kappa > 0$, as $t \downarrow 0$,

$$c(\kappa, t) \approx P(X_t > \kappa) = F_t(\kappa).$$

Since

$$c(\kappa, t) = E[(e^{X_t} - e^{\kappa})1\{X_t > \kappa\}] = E[e^{X_t} - e^{\kappa} \mid X_t > \kappa] P(X_t > \kappa)$$

In many (but not all) cases, the first factor $E[e^{X_t} - e^{\kappa} \mid X_t > \kappa]$ gives a negligible contribution with respect to $P(X_t > \kappa)$.
Crucial Observation

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In many (but not all) cases, the first factor $E[e^{X_t} - e^{\kappa} \mid X_t > \kappa]$ gives a negligible contribution with respect to $P(X_t > \kappa)$

We need to distinguish two regimes for $(\kappa, t)$:

- **typical deviations**: tail probability such that $\lim \overline{F}_t(\kappa) > 0$
- **atypical deviations**: tail probability vanishes $\overline{F}_t(\kappa) \to 0$
Atypical deviations \((\overline{F}_t(\kappa) \to 0)\): assumption

### Regular Decay

We consider a family \( (\kappa, t) = ((\kappa_s, t_s))_{s \geq 0} \) such that \( \forall \varrho \in [1, \infty) \) the following limit exists:

\[
I_+^+ (\varrho) := \lim \frac{\log \overline{F}_t (\varrho \kappa)}{\log \overline{F}_t (\kappa)},
\]

and moreover

\[
\lim_{\varrho \downarrow 1} I_+^+ (\varrho) = 1.
\]

Assumption can be checked in many concrete models

Equivalent to regular variation (Benaim&Friz) for fixed \( t > 0 \)
Atypical deviations ($\bar{F}_t(\kappa) \to 0$): assumption

**Regular Decay**

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$$l_+ (\rho) := \lim_{\rho \to 1} \frac{\log \bar{F}_t (\rho \kappa)}{\log \bar{F}_t (\kappa)} ,$$

and moreover

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- Assumption can be checked in many concrete models
- **Equivalent** to regular variation (Benaim&Friz) for fixed \(t > 0\)

\[
\log \mathcal{F}_t(\kappa) \sim L(\kappa) \kappa^\alpha
\]
Consider an arbitrary family of \((\kappa, t) = ((\kappa_s, t_s))_{s \geq 0}\) such that \(\overline{F}_t(\kappa) \to 0\) (atypical deviations).
Assume “Regular Decay” + suitable moment assumptions.
Theorem 2: tail probability to option price, atypical dev.

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Assume “Regular Decay” + suitable moment assumptions.

- Case of \(\lim \inf \kappa > 0, \lim \sup t < \infty\).

\[
\log c(\kappa, t) \sim \log \overline{F}_t(\kappa) + \kappa.
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- Case of \(\lim \inf \kappa > 0\), \(\lim \sup t < \infty\).
  \[
  \log c(\kappa, t) \sim \log \overline{F}_t(\kappa) + \kappa.
  \]

- Case of \(\kappa \to 0\) and \(t \to 0\).
  \[
  \log \left( \frac{c(\kappa, t)}{\kappa} \right) \sim \log \overline{F}_t(\kappa).
  \]
Corollary (tail probability to implied volatility, atypical dev.)

Under the previous assumption, if moreover \(-\log \overline{F}_t(\kappa)/\kappa \to \infty\),

\[
\sigma_{\text{imp}}(\kappa, t) \sim \frac{\kappa}{\sqrt{2t \left( -\log \overline{F}_t(\kappa) \right)}}.
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- Explicit link between right tail probability $\bar{F}_t(\kappa)$ and implied volatility $\sigma_{\text{imp}}(\kappa, t)$ can be applied to many concrete models.
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- Explicit link between right tail probability $\bar{F}_t(\kappa)$ and implied volatility $\sigma_{\text{imp}}(\kappa, t)$ can be applied to many concrete models.
- Analogous link between left tail probability $F_t(-\kappa)$ and implied volatility $\sigma_{\text{imp}}(-\kappa, t)$, under weaker assumptions!
Typical deviations \( \lim \bar{F}_t(\kappa) > 0 \)

**Assumption:** there is a positive function \( \gamma_t \to 0 \) such that

\[
\frac{X_t}{\gamma_t} \xrightarrow{d} Y.
\]

(In many stochastic volatility models \( \gamma_t = \sqrt{t} \) and \( Y = N(0, \sigma_0^2) \).)
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**Theorem**

Under suitable moment conditions

$$
c(a\gamma_t, t) \sim \gamma_t \ E[(Y - a)^+] ,
$$
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Theorem

Under suitable moment conditions

$$c(a\gamma_t, t) \sim \gamma_t \mathbb{E}[(Y - a)^+] \quad \text{and} \quad \sigma_{\text{imp}}(a\gamma_t, t) \sim C_Y(a) \frac{\gamma_t}{\sqrt{t}}.$$
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How to apply results

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How to apply results

- For **typical deviations** we study the **weak convergence** of $X_t$
- For **atypical deviations**
  - we study the **large deviations** of $X_t$
  - we need sharp asymptotics **only** for the logarithm of the tail probability (Gärtner-Ellis theorem)
Carr-Wu's Finite Logstable Model

The Carr & Wu Model is characterized by the following characteristic function:

$$E[e^{iuxt}] = \exp \left\{ t \left[ iu\mu - |u|^{\alpha} \sigma^{\alpha} \left( 1 + i \; \text{sign}(u) \tan \left( \frac{\pi \alpha}{2} \right) \right) \right] \right\}.$$
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Atypical deviations

If \( \frac{\kappa}{t^{1/\alpha}} \to \infty \) with \( 0 < t \leq T \), then

- **Right-wing asymptotics:** \( \sigma_{\text{imp}}(\kappa, t) \sim B_{\alpha} \left( \frac{\kappa}{t} \right)^{-\frac{2-\alpha}{2(\alpha-1)}} \),


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- **Left-wing asymptotics:**

\[ \sigma_{\text{imp}}(-\kappa, t) \sim \left( \sqrt{\frac{\log \frac{\kappa^\alpha}{t^\kappa}}{\kappa}} + 1 - \sqrt{\frac{\log \frac{\kappa^\alpha}{t^\kappa}}{\kappa}} \right) \sqrt{\frac{2\kappa^\alpha}{t^\kappa}}. \]
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Typical deviations

If $t \to 0$ then $\sigma_{\text{imp}}(at^{1/\alpha}, t) \sim C(a) t^{\frac{2-\alpha}{2\alpha}}$. 
Merton’s Jump diffusion model

In Merton’s Jump diffusion model the log-price evolves as

\[ X_t = \sigma W_t + \alpha t + \sum_{i=0}^{N_t} Y_i \]

with \( Y_i \sim N(\mu, \delta^2) \) and \( N_t \sim Pois(\lambda) \).
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**Atypical deviations**

For any family of \((\kappa, t)\) with \(\kappa \to \infty\), or when \(\kappa\) is fixed and \(t \to 0\) we have

\[ \sigma_{imp}^2(\kappa, t) \sim \frac{\kappa}{2t} \frac{\delta}{\sqrt{2 \log \frac{\kappa}{t}}} \]
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Atypical deviations

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Typical deviations

If \( t \to 0 \) then \( \sigma_{\text{imp}}(a\sqrt{t}, t) = C(a) \).
The Heston Model

\[
\begin{aligned}
\begin{cases}
    \frac{dS_t}{S_t} &= \sqrt{V_t} \, dW^1_t, \\
    \frac{dV_t}{V_t} &= -\lambda (V_t - \nu) \, dt + \eta \sqrt{V_t} \, dW^2_t, \\
    X_0 &= 0, \quad V_0 = \sigma_0, \quad d\langle W^1, W^2 \rangle_t = \rho \, dt
\end{cases}
\end{aligned}
\]
The Heston Model

\[
\begin{align*}
    dS_t &= S_t \sqrt{V_t} \, dW^1_t, \\
    dV_t &= -\lambda (V_t - \vartheta) \, dt + \eta \sqrt{V_t} \, dW^2_t, \\
    X_0 &= 0, \quad V_0 = \sigma_0, \quad d\langle W^1, W^2 \rangle_t = \varrho \, dt
\end{align*}
\]

We define

\[
p^*(t) := \sup\{p > 0 : \mathbb{E}[S_t^p] < \infty\}.
\]

If \(\varrho > -1\), as \(t \to 0\)

\[
p^*(t) \sim \frac{C}{t}.
\]
The Heston Model - Implied Volatility

\( \kappa \to +\infty, \ t \ \text{fixed} - \text{Andersen & Piterbarg (2007)} \)

\[
\sigma_{\text{imp}}(\kappa, t) \sim \frac{\sqrt{2\kappa}}{\sqrt{t}} \left( \sqrt{p^*(t)} - \sqrt{p^*(t) - 1} \right).
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\]

\( \kappa \ fixed, \ t \downarrow 0 - \text{Forde & Jacquier (2009)} \)

\[
\sigma_{\text{imp}}(\kappa, t) \sim \frac{\kappa}{\sqrt{2 \Lambda^*(\kappa)}}.
\]
The Heston Model - Implied Volatility

\( \kappa \to +\infty, \ t \ fixed - \text{Andersen \& Piterbarg (2007)} \)

\[
\sigma_{\text{imp}}(\kappa, t) \sim_{k \uparrow \infty} \frac{\sqrt{2\kappa}}{\sqrt{t}} \left( \sqrt{p^*(t)} - \sqrt{p^*(t) - 1} \right).
\]

\( \kappa \ fixed, \ t \downarrow 0 - \text{Forde \& Jacquier (2009)} \)

\[
\sigma_{\text{imp}}(\kappa, t) \sim_{t \downarrow 0} \frac{\kappa}{\sqrt{2 \Lambda^*(\kappa)}}.
\]

\( \kappa \to +\infty, \ t \downarrow 0 \ (\text{Conjecture}) \)

\[
\sigma_{\text{imp}}(\kappa, t) \sim_{t \downarrow 0, \kappa \to \infty} \frac{\kappa}{\sqrt{2 C}}.
\]
Thank you for your attention
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