A hybrid tree-finite difference approach for the Heston model

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From a joint work with:
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We propose a mixed tree-finite difference method in order to approximate the Heston model.

We prove the convergence by embedding the procedure in a bivariate Markov chain. We also study the convergence of European and American option prices.

We provide numerical results that give accurate European and American option prices in the Heston model, showing the reliability and the efficiency of the algorithm.

We show how to generalize the procedure (tree+finite differences) to the Bates model, the Heston-Hull-White model and the Heston-Hull-White2D model.
Previous tree and finite difference literature

Tree methods for the Heston model:


Finite differences for the 2D Heston PDE:


The Heston model

Under the risk neutral measure, the pair \((S, V)\) of the share price and the volatility process solves the SDE

\[
\frac{dS_t}{S_t} = (r - \delta)dt + \sqrt{V_t} \, dZ_S(t), \quad S(0) = S_0 > 0
\]

\[
dV_t = \kappa(\theta - V(t))dt + \sigma \sqrt{V_t} \, dZ_V(t), \quad V(0) = V_0 > 0
\]

where:

- \(r\) and \(\delta\) are the risk free interest rate and the continuous dividend rate respectively,
- \(\sigma\) is the volatility of the volatility,
- \(\kappa\) is the reversion speed,
- \(\theta\) is the long run variance,
- \(Z_S\) and \(Z_V\) are correlated Brownian motions:

\[
d\langle Z_S, Z_V \rangle_t = \rho dt, \quad |\rho| < 1.
\]
Take $N$ “large” and $h = T/N$. For $n = 0, 1, \ldots, N$, consider the lattice used in Appolloni, Caramellino, Zanette (2014)

$$V^h_n = \{v_{n,k}\}_{k=0,1,\ldots,n} \quad \text{with}$$

$$v_{n,k} = \left(\sqrt{V_0 + \frac{\sigma}{2}(2k-n)\sqrt{h}}\right)^2 1_{\sqrt{V_0 + \frac{\sigma}{2}(2k-n)\sqrt{h}}>0}$$

We define the multiple jumps

$$k^h_d(n, k) = \max\{k^* : 0 \leq k^* \leq k \text{ and } v_{n,k} + \mu \mathbb{V}(v_{n,k})h \geq v_{n+1,k^*}\},$$

$$k^h_u(n, k) = \min\{k^* : k + 1 \leq k^* \leq n + 1 \text{ and } v_{n,k} + \mu \mathbb{V}(v_{n,k})h \leq v_{n+1,k^*}\}$$

in which $\mu \mathbb{V}$ denotes the drift coefficient of $\mathbb{V}$. 
The robust tree method for $V$

Figure: *Standard jumps and multiple jumps for the discrete approximation of the process $V$.*/
The robust tree method for $V$

Starting from the node $(n, k)$, the discrete process can reach the up-jump node $(n + 1, k^h_{u}(n, k))$ or the down-jump node $(n + 1, k^h_{d}(n, k))$ with transition probability:

- **up-jump:** $p^h_{k^h_{u}(n, k)} = 0 \lor \frac{\mu V(v_{n,k}) h + v_{n,k} - v_{n+1,k^h_{u}(n,k)}}{v_{n+1,k^h_{u}(n,k)} - v_{n+1,k^h_{d}(n,k)}} \land 1,$
- **down-jump:** $p^h_{k^h_{d}(n, k)} = 1 - p^h_{k^h_{u}(n, k)}.$

Multiple jumps & jump probabilities are set in order to match the first local moment of the tree and of the process $V$ up to order 1 w.r.t. $h$. As a consequence, as $h \to 0$ one gets weak convergence on the path space.

**Remark.** In order to obtain the convergence, we do not need to require the Feller condition $2\kappa \theta \geq \sigma^2.$
The transformed process \( Y \)

We consider the diffusion pair \((Y, V)\), where

\[
Y_t = \log S_t - \frac{\rho}{\sigma} V_t.
\]

Set: \( Z_V = W \) and \( Z_S = \rho W + \bar{\rho} Z \), with \( \bar{\rho} = \sqrt{1 - \rho^2} \) and \((W, Z)\) standard Brownian motion in \( \mathbb{R}^2 \). Then,

\[
\begin{align*}
dY_t &= \left( r - \delta - \frac{1}{2} V_t - \frac{\rho}{\sigma} \kappa (\theta - V_t) \right) dt + \bar{\rho} \sqrt{V_t} \, dZ_t, \\
dV_t &= \kappa (\theta - V_t) dt + \sigma \sqrt{V_t} \, dW_t,
\end{align*}
\]

with \( Y_0 = \log S_0 - \frac{\rho}{\sigma} V_0 \). In the following, we set \( \mu_Y \) and \( \mu_V \) the drift coefficient of \( Y_t \) and \( V_t \) respectively:

\[
\mu_Y(v) = r - \delta - \frac{1}{2} v - \frac{\rho}{\sigma_v} \kappa (\theta - v) \quad \text{and} \quad \mu_V(v) = \kappa (\theta - v).
\]
The approximation of $Y$

Let $\bar{V}^h = (\bar{V}^h_n)_{n=0,\ldots,N}$ denote the tree process approximating $V$ and set $V^h_t = \bar{V}^h_{\lfloor t/h \rfloor}$, $t \in [0, T]$, the associated piecewise constant and càdlàg approximating path.

In order to approximate $Y$, we construct a Markov chain from the finite difference method.

We start from the Euler scheme: $Y^h_0 = Y_0$ and for $t \in (nh, (n + 1)h]$, $n = 0, \ldots, N$, set

$$Y^h_t = Y^h_{nh} + \mu_Y(V^h_{nh})(t - nh) + \bar{\rho}\sqrt{V^h_{nh}}(Z_t - Z_{nh}),$$

$Z$ being independent of the noise driving $\bar{V}^h$. 

Let $f$ be a suitable function depending on both variables $(y, v)$. Then,

$$
\mathbb{E}(f(Y_{(n+1)h}, V_{(n+1)h}) \mid Y_{nh} = y, V_{nh} = v) \\
\simeq \mathbb{E}(f(Y_{(n+1)h}^h, V_{(n+1)h}^h) \mid Y_{nh}^h = y, V_{nh}^h = v) \\
= \mathbb{E}(u^h(nh, y; v, V_{(n+1)h}^h) \mid V_{nh}^h = v)
$$

where

$$u^h(nh, y; v, z) = \mathbb{E}(f(Y_{(n+1)h}^h, z) \mid Y_{nh}^h = y, V_{nh}^h = v).$$

and

$$u^h(nh, y; v, z) = u^h(s, y; v, z)|_{s=nh}$$
The approximation of the pair \((Y, V)\)

Main fact: \((s, y) \mapsto u^h(s, y; v, z)\) solves the PDE

\[
\begin{align*}
&\partial_s u^h + \mu Y(v) \partial_y u^h + \frac{1}{2} \bar{\rho}^2 v \partial_{yy} u^h = 0, \quad y \in \mathbb{R}, \; nh < s < (n+1)h, \\
u^h((n+1)h, y; v, z) = f(y, z), \quad y \in \mathbb{R}.
\end{align*}
\] (3)

\(\Rightarrow\) simple problem: one dimensional, constant coefficients.
\(\Rightarrow\) \(u^h(s, y; v, z)\) at \(s = nh\) can be numerically found by using a one-step finite difference method.
In practice we consider the finite grid $\mathcal{Y}^h = \{y_j\}_{j \in \mathcal{J}_{M_h}}$ with equally spaced points

$$y_j = Y_0 + j \Delta y_h, \quad j \in \mathcal{J}_{M_h} = \{-M_h, \ldots, M_h\}.$$ 

The approximation of $u^h(nh, y; v, z)$ is done by adding to (3) suitable boundary conditions - we use Neumann type conditions (but others can be chosen).
The finite difference scheme

The behavior of the solution of problem (3) changes with respect to the magnitude of the rate between the diffusion coefficient ($\rho^2 v/2$) and the advection term ($\mu \gamma(v)$).

We fix a small real threshold $\epsilon_h > 0$. Then:

- case $v > \epsilon_h$: we use an implicit scheme

$$\frac{u_j^{n+1} - u_j^n}{h} + \mu \gamma(v) \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta y} + \frac{1}{2} \rho^2 v \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta y^2} = 0,$$

with boundary conditions:

$$u_{-M-1}^n = u_{-M+1}^n, \quad u_{M+1}^n = u_{M-1}^n;$$
The finite difference scheme

- case $\nu < \epsilon_h$: we use an explicit scheme:
  * if $\mu_Y(\nu) \geq 0$:

$$
\frac{u_{j+1}^{n+1} - u_j^n}{h} + \mu_Y(\nu) \frac{u_{j+1}^{n+1} - u_j^{n+1}}{\Delta y} + \frac{1}{2} \bar{\rho}^2 \nu \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta y^2} = 0;
$$

  * if $\mu_Y(\nu) < 0$:

$$
\frac{u_{j+1}^{n+1} - u_j^n}{h} + \mu_Y(\nu) \frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{\Delta y} + \frac{1}{2} \bar{\rho}^2 \nu \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta y^2} = 0;
$$

here, the boundary conditions are

$$
u_{-M-1}^{n+1} = v_{-M+1}^{n+1}, \quad v_{M+1}^{n+1} = v_{M-1}^{n+1}.$$
We obtain a finite-dimensional matrix $\Pi^h(\nu)$ giving the solution from the finite difference approach. By resuming, we get

$$E(f(Y_{(n+1)h}, V_{(n+1)h}) \mid Y_{nh} = y_i, V_{nh} = \nu) \approx \sum_{j \in \mathcal{J}_{Mh}} \Pi^h(\nu)_{i,j} E(f(y_j, V^h_{(n+1)h}) \mid V^h_{nh} = \nu), \quad i \in \mathcal{J}_{Mh}.$$ 

**Theorem**

There exists $h_0$ (quantitative estimates for it!) such that for every $h < h_0$ and $\nu \in \bigcup_{n=0}^{N} V_n^h$ then $\Pi^h(\nu) = (\Pi^h(\nu)_{i,j})_{i,j \in \mathcal{J}_{Mh}}$ is well defined and moreover, it is a stochastic matrix.
2-dimensional Markov chain

We take $h$ small and we call $\bar{X}^h = (\bar{X}^h_n)_{n=0,1,...,N}$ the 2-dimensional Markov chain with transition probability law

$$
\mu^h(y_j, v_{n+1}, k^* \mid y_i, v_{n,k}) = \begin{cases} 
\prod^h(v_{n,k})_{ij} p^h_{k^*_u(n,k)} & \text{if } k^* = k^ h_u(n,k) \\
\prod^h(v_{n,k})_{ij} p^h_{k^*_d(n,k)} & \text{if } k^* = k^ h_d(n,k) \\
0 & \text{otherwise,}
\end{cases}
$$

for every $(y_i, v_{n,k}) \in \mathcal{Y}^h \times \mathcal{V}^h_n$ and $(y_j, v_{n+1,k^*}) \in \mathcal{Y}^h \times \mathcal{V}^h_{n+1}$.
Then, for \( n = 0, 1, \ldots, N - 1, \) \( y_i \in \mathcal{Y}^h \) and \( v_{n,k} \in \mathcal{V}_n^h \) we get:

\[
\mathbb{E}(f(Y_{n+1}^h, V_{n+1}^h) \mid Y_{nh} = y_i, V_{nh} = v_{n,k}) = \mathbb{E}(f(\tilde{X}_{n+1}^h) \mid \tilde{X}_n^h = (y_i, v_{n,k})) \\
= \sum_{k^*, j} \Pi^h(v_{n,k})_{i,j} p^h_{k^*} f(y_j, v_{n+1,k^*}),
\]

the above sum running on \( k^* \in \{k_u^h(n, k), k_d^h(n, k)\} \) and \( j \in \mathcal{J}_{M_h} \).

\( \Rightarrow \) we have constructed an approximation of \((Y, V)\) at times \( t_n = nh, \) \( n = 0, 1, \ldots, N, \) through the Markov chain \((\tilde{X}_n^h)_{n=0,1,\ldots,N}\).

Before studying the convergence, we briefly discuss how to use the above procedure to numerically price American options.
American option pricing

Consider an American option with maturity $T$ and payoff function

$$\Psi(Y_t, V_t) = \Phi(e^{Y_t - \frac{\rho}{\sigma} V_t}), \quad t \in [0, T].$$

By considering the discrete dynamic programming principle and by using the approximation of $(Y, V)$ through the Markov chain $\bar{X}_n^h$, $n = 0, 1, \ldots, N$, we approximate the price as follows:

for $n = 0, 1, \ldots, N$, we define $\tilde{P}_h(nh, y, v)$ for $(y, v) \in \mathcal{Y}^h \times \mathcal{V}_n^h$ by

$$
\begin{cases}
\tilde{P}_h(T, y_i, v_{N,k}) = \Psi(y_i, v_{N,k}) & i \in J_M^h \text{ and } v_{N,k} \in \mathcal{V}_n^h \\
\tilde{P}_h(nh, y_i, v_{n,k}) = \max \left\{ \Psi(y_i, v_{n,k}), e^{-rh} \times \right. \\
\left. \sum_{k^*, j} \Pi^h(v_{n,k})_{i,j} \tilde{P}_h((n + 1)h, y_j, v_{n+1,k^*}) p_{k^*}^h \right\}, \\
i \in J_M^h \text{ and } v_{n,k} \in \mathcal{V}_n^h.
\end{cases}
$$
We set up the dependence on the time step $h$ for the space-step $\Delta y_h$, the number $M_h$ giving the points of the grid $\mathcal{Y}_{M_h}$ and the threshold $\epsilon_h$ that allows us to use the explicit or the implicit finite difference scheme:

$$\Delta y_h = c_y h^p, \quad M_h = c_M h^{-q}, \quad \epsilon_h = c_\epsilon h^p$$  \hspace{1cm} (4)

where $c_M > 0$ and the constants $c_y, c_\epsilon, p, q > 0$ are chosen as follows

\begin{align*}
    p < 1, \quad q > p, \quad &\frac{2c_y}{\rho^2} |r - \delta - \frac{\rho}{\sigma_v} \kappa \theta| < c_\epsilon, \quad \text{or} \quad \\
    p = 1, \quad q > p, \quad &\frac{2c_y}{\rho^2} |r - \delta - \frac{\rho}{\sigma_v} \kappa \theta| < c_\epsilon < \left(\frac{1}{2} - \frac{1}{c_y} |r - \delta - \frac{\rho}{\sigma_v} \kappa \theta|\right) \frac{c_y^2}{\rho^2},
\end{align*}

(5)
We set $D([0, T]; \mathbb{R}^2)$ the space of the $\mathbb{R}^2$-valued and càdlàg functions on the interval $[0, T]$, that we assume to be endowed with the Skorohod topology.

We set $X^h$ as the process in $D([0, T]; \mathbb{R}^2)$ given by the piecewise constant and càdlàg interpolation in time of $\bar{X}^h$:

$$X^h_t = \bar{X}^h_{\lfloor t/h \rfloor}, \quad t \in [0, T].$$

**Theorem**

Suppose that (4) and (5) hold. Then as $h \to 0$, the sequence $\{X^h\}_h$ weakly converges in the space $D([0, T]; \mathbb{R}^2)$ to the diffusion process $X = (Y, V)$ solution to (1)-(2).
Convergence of option prices

Consider a European option with payoff function $f : \mathcal{D}([0, T]; \mathbb{R}) \to \mathbb{R}_+$. Set the transformed payoff-function

$$g(y, v) = f(e^{y} + \frac{\rho}{\sigma}v), \quad (y, v) \in \mathcal{D}([0, T]; \mathbb{R}^2).$$

The associated option prices on the continuous and the discrete model as seen at time 0 are given by

$$P_{Eu} = \mathbb{E}(\tilde{g}(Y, V)) \quad \text{and} \quad P_{Eu}^h = \mathbb{E}(\tilde{g}(Y^h, V^h)),$$

respectively, $\tilde{g}$ denoting the discounted payoff, i.e. $\tilde{g} = e^{-rT}g$.
Suppose that \((y, \nu) \mapsto \tilde{g}(y, \nu)\) is continuous and there exists \(a > 0\) and \(h_* > 0\) such that

\[
\sup_{h < h_*} \mathbb{E}(|\tilde{g}(Y^h, V^h)|^{1+a}) < \infty.
\]

Then, \(P_{Eu}^h \to P_{Eu}\) as \(h \to 0\).

As for American style options, even for simple payoffs things are more difficult because of the presence of optimal stopping times. However, under conditions similar to the above ones, the techniques in Amin and Khanna ’94 can be adapted, and the convergence of the prices holds.

As a consequence, the price of European and American put options evaluated on our discrete process converges to the true price.
Preliminary results (work in progress!) for European options only.

- If the payoff function ensures that the solution of the 2D Heston PDE exists and is smooth enough: the speed of convergence is $h$.

- For quite general payoff functions: the convergence holds to the 2D Heston PDE viscosity solution (but we have not, up to now, estimates for the speed of convergence).
Numerical results: vanilla options

- We compare the performance of the hybrid tree-finite difference algorithm with the closed formula (European case) or with the tree method of Vellekoop and Nieuwenhuis (American case).
- Parameters: $S_0 = 100$, $K = 100$, $T = 1$, $r = \log(1.1)$, $\delta = 0$, $V_0 = 0.1$, $\theta = 0.1$, $\kappa = 2$ and $\rho = -0.5$.
- In order to study the numerical robustness of the algorithms we choose different values for $\sigma$: 0.04, 0.5, 1. For $\sigma = 1$ the Feller condition $2\kappa\theta \geq \sigma^2$ is not satisfied.
- HTFD1: fixed number of time steps $N_t = 100$ and varying number of space steps $N_S = 50, 100, 200, 400$;
- HTFD2: number of time steps equal to the number of space steps: $N_t = N_S = 50, 100, 200, 400$. 
## Numerical results: European options

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**Table:** Prices of European put options. $\sigma = 0.04, 0.5, 1$. $S_0 = 100$, $K = 100$, $T = 1$, $r = \log(1.1)$, $\delta = 0$, $V_0 = 0.1$, $\theta = 0.1$, $\kappa = 2$, $\rho = -0.5$. 
### Numerical results: American options

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**Table:** Prices of American put options. $\sigma = 0.04, 0.5, 1$. $S_0 = 100$, $K = 100$, $T = 1$, $r = \log(1.1)$, $\delta = 0$, $V_0 = 0.1$, $\theta = 0.1$, $\kappa = 2$, $\rho = -0.5$. 
### Numerical results: computational time

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</table>

**Table:** *Computational times (in seconds) for European put options for $\sigma = 0.5$.***
We study continuously monitored barrier options and we compare our hybrid tree-finite difference algorithm with the numerical results of the method of lines MOL of Chiarella et al.

We consider European and American up-and-out call options with the following set of parameters: $K = 100$, $T = 0.5$, $r = 0.03$, $\delta = 0.05$, $V_0 = 0.1$, $\theta = 0.1$, $\kappa = 2$, $\rho = -0.5$.

The up barrier is $H = 130$. We choose different values for $S_0$: $S_0 = 80, 100, 120$. 
Numerical results: European barrier options

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>$N_S$</th>
<th>HTFD1</th>
<th>HTFD2</th>
<th>MOL</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>50</td>
<td>0.913861</td>
<td>0.875374</td>
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</tr>
<tr>
<td></td>
<td>100</td>
<td>0.893484</td>
<td>0.893484</td>
<td></td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>0.895127</td>
<td>0.900893</td>
<td>0.9029</td>
</tr>
<tr>
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<td>400</td>
<td>0.897820</td>
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<td></td>
</tr>
<tr>
<td>100</td>
<td>50</td>
<td>2.635396</td>
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<td>100</td>
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<td>2.606249</td>
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<tr>
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<td>200</td>
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<td>2.591857</td>
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<td>400</td>
<td>2.603679</td>
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</tr>
<tr>
<td>120</td>
<td>50</td>
<td>1.417225</td>
<td>1.438429</td>
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<tr>
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<td>1.485704</td>
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<td>400</td>
<td>1.504755</td>
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Table: Prices of European call up-and-out options. Up barrier is $H = 130$. $K = 100$, $T = 0.5$, $r = 0.03$, $\delta = 0.05$, $V_0 = 0.1$, $\theta = 0.1$, $\kappa = 2$, $\rho = -0.5$. 
### Numerical results: American barrier options

<table>
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<tr>
<th>$S_0$</th>
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<th>HTFD2</th>
<th>MOL</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1.285959</td>
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<td>100</td>
<td>1.369914</td>
<td>1.369914</td>
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<td>1.396628</td>
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<td>8.286667</td>
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<tr>
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<tr>
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<td>21.779648</td>
<td>21.804518</td>
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</table>

**Table:** Prices of American call up-and-out options. Up barrier is $H = 130$. $K = 100$, $T = 0.5$, $r = 0.03$, $\delta = 0.05$, $V_0 = 0.1$, $\theta = 0.1$, $\kappa = 2$, $\rho = -0.5$. 
Introduction
The hybrid tree-finite difference approach
Convergence results
Numerical results
Generalization to other models

Numerical results: computational time

<table>
<thead>
<tr>
<th>$N_S$</th>
<th>HTFD1</th>
<th>HTDF2</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
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<td>0.017</td>
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<td>100</td>
<td>0.132</td>
<td>0.132</td>
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<tr>
<td>200</td>
<td>0.284</td>
<td>1.079</td>
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<tr>
<td>400</td>
<td>0.535</td>
<td>8.901</td>
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**Table:** Computational times (in seconds) for European Barrier options.
Bates model

The Bates model differs from the Heston model in the presence of jumps in the equation for the share price $S$:

$$\frac{dS_t}{S_t} = (r - \delta)dt + \sqrt{V_t} dZ_S(t) + dN_t, \quad S(0) = S_0 > 0$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma \sqrt{V_t} dZ_V(t), \quad V(0) = V_0 > 0.$$

Here, $N_t$ is a compound Poisson process independent of the correlated Brownian motion $(Z_S, Z_V)$ with intensity $\lambda$ and independent jumps $J_1, J_2, \ldots$ whose common law is

$$\log(1 + J) \sim N\left(\log(1 + \gamma) - \frac{1}{2}\alpha^2, \alpha^2\right).$$

So, the PDE problem becomes a PIDE problem.
Bates model

Numerical results:

- **HTFD1**: fixed number of time steps $N_t = 50$ and varying number of space steps $N_S = 353, 704, 1406, 2811$;
- **HTFD2**: fixed number of time steps $N_t = 100$ and varying number of space steps $N_S = 353, 704, 1406, 2811$. 
Numerical results: Bates model

<table>
<thead>
<tr>
<th>$\rho = 0.5$</th>
<th>$N_S$</th>
<th>HTFD1</th>
<th>HTFD2</th>
<th>MOL</th>
<th>PSOR</th>
</tr>
</thead>
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<tr>
<td>$S_0 = 80$</td>
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<td>1.481645</td>
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<tr>
<td></td>
<td>704</td>
<td>1.480856</td>
<td>1.483010</td>
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<tr>
<td></td>
<td>1406</td>
<td>1.480720</td>
<td>1.482807</td>
<td>1.4843</td>
<td>1.4848</td>
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<td>2811</td>
<td>1.480677</td>
<td>1.482761</td>
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<tr>
<td>$S_0 = 100$</td>
<td>353</td>
<td>7.704426</td>
<td>7.705136</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>704</td>
<td>7.703646</td>
<td>7.703875</td>
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<tr>
<td></td>
<td>1406</td>
<td>7.703257</td>
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<td>$S_0 = 120$</td>
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<td>21.365526</td>
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<tr>
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<td>2811</td>
<td>21.365180</td>
<td>21.365366</td>
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</tr>
</tbody>
</table>

Table: Prices of American call options. $K = 100$, $T = 0.5$, $r = 0.03$, $\delta = 0.05$, $V_0 = 0.04$, $\theta = 0.04$, $\kappa = 2$, $\sigma = 0.4$, $\lambda = 5$, $\gamma = 0$, $\alpha = 0.1$, $\rho = 0.5$. 
Numerical results: Bates model

<table>
<thead>
<tr>
<th>$N_S$</th>
<th>HTFD1</th>
<th>HTDF2</th>
</tr>
</thead>
<tbody>
<tr>
<td>353</td>
<td>0.52</td>
<td>2.07</td>
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<tr>
<td>704</td>
<td>1.07</td>
<td>4.18</td>
</tr>
<tr>
<td>1406</td>
<td>2.36</td>
<td>9.28</td>
</tr>
<tr>
<td>2811</td>
<td>4.86</td>
<td>19.37</td>
</tr>
</tbody>
</table>

Table: Computational times (in seconds) for American Call options for $S_0 = 100$, $\rho = -0.5$. 
The model is a Heston model with a stochastic interest rate $r$: under the risk neutral measure, the model is

$$\frac{dS_t}{S_t} = (r_t - \delta)dt + \sqrt{V_t} dZ_t, \quad S_0 > 0,$$

$$dV_t = \kappa_V (\theta_V - V_t)dt + \sigma_V \sqrt{V_t} dW^1_t, \quad V_0 > 0$$

$$dr_t = \kappa_r (\theta_r(t) - r_t)dt + \sigma_r dW^2_t, \quad r_0 > 0.$$

where $Z$, $W^1$ and $W^2$ are Brownian motions with correlations

$$d\langle Z, W^1 \rangle_t = \rho_1 dt, \quad d\langle Z, W^2 \rangle_t = \rho_2 dt, \quad d\langle W^1, W^2 \rangle_t = 0.$$

Here, $r$ follows a generalized OU process: $\theta_r$ is a (deterministic) function determined by the market values of the zero-coupon bonds.
The procedure:

- a 2-dimensional binomial tree for the pair \((V, r)\);
- a finite difference approach in the \(S\)-direction.

Numerical results: work in progress!
Here, $\delta$ is stochastic as well:

\[
\frac{dS_t}{S_t} = (r_t - \delta_t) dt + \sqrt{V_t} dZ_t, \quad S_0 > 0,
\]
\[
dV_t = \kappa_V (\theta_V - V_t) dt + \sigma_V \sqrt{V_t} dW^1_t, \quad V_0 > 0,
\]
\[
dr_t = \kappa_r (\theta_r(t) - r_t) dt + \sigma_r dW^2_t, \quad r_0 > 0,
\]
\[
d\delta_t = \kappa_\delta (\theta_\delta(t) - \delta_t) dt + \sigma_\delta dW^3_t, \quad \eta_0 > 0,
\]

where $Z$, $W^1$, $W^2$ and $W^3$ are Brownian motions with correlations

\[
d\langle Z, W^i \rangle_t = \rho_i \ dt \quad i = 1, 2, 3, \quad d\langle W^i, W^j \rangle_t = 0 \quad i \neq j.
\]

$\delta$ follows a generalized OU process.
The procedure:

- a 3-dimensional binomial tree for the triple \((V, r, \delta)\);
- a finite difference approach in the \(S\)-direction.

Numerical results: work in progress!
References


